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Notes on tolerance relations of lattices

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0. Introduction

A *tolerance relation* Θ on a lattice L is a reflexive and symmetric binary relation satisfying the substitution property. In 1982, G. CZÉDLI [1] proved that, for a lattice L and a tolerance relation Θ, the maximal Θ-connected subsets of L form a lattice. He considered lattices as algebras of type $\langle 2, 2\rangle$ and gave an algebraic proof. In Section 1, we investigate tolerances from the point of view of partial ordering in detail; in particular, we give an order-theoretical proof of Czédli's result. Our proof avoids Zorn's axiom needed by Czédli. Some results on Θ-block fixing sets and consequences thereof are added.

Tolerances can be viewed as quotients of congruences in a natural way. Using this fact, we extend the Second Isomorphism Theorem from congruences to tolerance relations in Section 2. In connection with the extended Second Isomorphism Theorem, a question on the product of lattice varieties arises naturally. In Section 3 we answer it partially and illustrate the situation with examples.

1. The lattice L/Θ

For concepts and notations not defined in this paper, see G. GRÄTZER [3]. Let $L=(L; \leqq)$ be a lattice and Θ a tolerance relation on L. $x\Theta y$ $(x, y\in L)$ denotes, as usual, that $(x, y)\in\Theta$ holds; also, $H_1\Theta H_2$ $(H_1, H_2\subseteqq L)$ denotes that $x\Theta y$ holds for every $x\in H_1$, $y\in H_2$. The following two lemmata are useful in many situations.

Lemma 1. *Let $x\leqq x'$ and $y\leqq y'$ be elements of L with $x'\Theta x$, $y'\Theta y$, $x\Theta y'$, $y\Theta x'$. Then $(x'\vee y')\Theta(x\wedge y)$.*

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Proof. $x\Theta y'$, $x\Theta x'$ imply that $x\Theta(x'\vee y')$. Similarly, $y\Theta(x'\vee y')$ holds. Thus, $(x\wedge y)\Theta(x'\vee y')$.

Lemma 2. *Let* x, y, x', y' *be elements of* L *with* $x\Theta x'$, $y\Theta y'$ *and* $x, y \leqq x'\wedge y'$. *Then* $x\Theta y$ *and* $x'\Theta y'$.

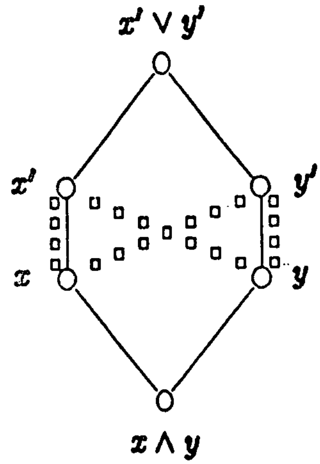

Diagram 1

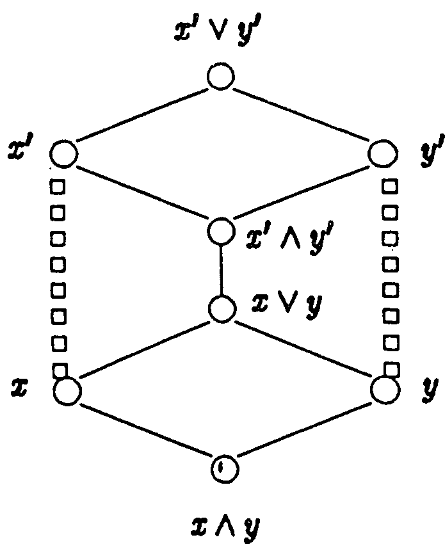

Diagram 2

Proof. $x\Theta x', y\Theta y'$ imply that $(x\vee y)\Theta(x'\vee y')$. From $x\vee y\leqq x', y'\leqq x'\vee y'$, we conclude that $x'\Theta y'$. The other assertion follows analogously.

We use the following notations and terminology. A subset H of L is called *Θ-connected* if $x\Theta y$ holds for all $x, y\in H$. If H is an arbitrary subset of L, then we define $C_H := \{x\in L;\ x\Theta h \text{ for all } h\in H\}$. C_H is either empty or it is a convex sublattice of L. C_H is not necessarily Θ-connected. We further define

$$(H] := \{x\in L;\ x\leqq h \text{ for some } h\in H\} \quad\text{and}\quad [H) := \{x\in L;\ x\geqq h \text{ for some } h\in H\}.$$

Finally, $H_\Theta := C_H\cap(H]$ and $H^\Theta := C_H\cap[H)$.

Lemma 3. *Let H be a subset of the lattice L.*

(1) *H_Θ is a Θ-connected, convex, $\wedge$-closed subset of L. If H is upward directed, then H_Θ is either empty or it is a sublattice of L.*

(2) *H^Θ is a Θ-connected, convex, $\vee$-closed subset of L. If H is downward directed, then H^Θ is either empty or it is a sublattice of L.*

Proof. We only prove (1). H_Θ is clearly convex and $\wedge$-closed. To show that it is Θ-connected, let $x, y\in H_\Theta$. There are $x', y'\in H$ with $x\leqq x'$, $y\leqq y'$, $x\Theta x'$, $y\Theta x'$, $x\Theta y'$, $y\Theta y'$. By Lemma 1, $x\Theta y$. If H is upward directed, we can choose $x'=y'$ and obtain $x\vee y\leqq x'$, hence $x\vee y\in(H]$. Since $x\vee y\in C_H$ is clear, we get $x\vee y\in H_\Theta$.

Lemma 4. *Let H be a Θ-connected subset of L.*

(1) *$(H^\Theta)^\Theta = H^\Theta$ and $(H_\Theta)_\Theta = H_\Theta$.*

(2) *$H\subseteqq H_\Theta\subseteqq (H_\Theta)^\Theta = ((H_\Theta)^\Theta)_\Theta = \dots$, and $(H_\Theta)^\Theta$ is a Θ-connected, convex sublattice of L, if $H\neq\emptyset$.*

Proof. (1) is clear. As to (2): Lemma 3 yields that $H\subseteqq H_\Theta\subseteqq (H_\Theta)^\Theta\subseteqq ((H_\Theta)^\Theta)_\Theta$. Let $x\in((H_\Theta)^\Theta)_\Theta$. Then $x\leqq y\geqq u\leqq v$ for some $y\in(H_\Theta)^\Theta$, $u\in H_\Theta$ and $v\in H$. We claim that $x\wedge u\in H_\Theta$. Indeed, clearly, $x\wedge u\leqq v$, hence $x\wedge u\in(H]$. If $h\in H$, then $x\Theta h$. Together with $u\Theta h$ we get $(x\wedge u)\Theta h$; hence, $x\wedge u\in C_H$ and $x\wedge u\in H_\Theta$, as claimed. Now $x\in[H_\Theta)\cap C_{H_\Theta} = (H_\Theta)^\Theta$, and the first part of (2) has been proved. H_Θ is Θ-connected and $\wedge$-closed, by Lemma 3. Hence, again by Lemma 3, $(H_\Theta)^\Theta$ is a Θ-connected, convex sublattice of L.

The significance of $(H_\Theta)^\Theta$ comes from the next lemma.

Lemma 5. *Let X be a subset of L. The following two statements are equivalent:*

(1) *X is a maximal Θ-connected subset of L.*

(2) *$X=(H_\Theta)^\Theta$ for some non-empty Θ-connected $H\subseteqq L$.*

Proof. (1) implies (2) follows by taking $H=X$ and by Lemma 4. In order to prove that (2) implies (1), we choose $u\in L$ with $u\Theta(H_\Theta)^\Theta$. For every $x\in(H_\Theta)^\Theta$, we get $u\wedge x\in((H_\Theta)^\Theta)_\Theta$. From $u\Theta((H_\Theta)^\Theta)_\Theta$, we get $u\in(((H_\Theta)^\Theta)_\Theta)^\Theta=(H_\Theta)^\Theta$. Hence, $(H_\Theta)^\Theta$ is a maximal Θ-connected subset of L.

In view of the last lemma, we call subsets of the form $(H_\Theta)^\Theta$ for Θ-connected subsets H of L *Θ-blocks of L*. The Θ-blocks are convex sublattices of L. They enjoy a useful property with respect to two natural preorderings on L. In order to prove it, we use the next trivial lemma.

Lemma 6. *For* $A, B\subseteq L$, *define*

$$A\mathbf{\vee}B := \{a\vee b;\ a\in A,\ b\in B\}$$

and

$$A\mathbf{\wedge}B := \{a\wedge b;\ a\in A,\ b\in B\}.$$

If A and B are Θ-connected, then so are $A\mathbf{\vee}B$ and $A\mathbf{\wedge}B$.

Definition 1. For $A, B\subseteq L$ we define the following three binary relations:
(1) $A\circ\leqq B :\Leftrightarrow$ For all $b\in B$ there is an $a\in A$ with $a\leqq b$.
(2) $A\leqq\circ B :\Leftrightarrow$ For all $a\in A$ there is a $b\in B$ with $a\leqq b$.
(3) $A\leqq B :\Leftrightarrow A\circ\leqq B$ and $A\leqq\circ B$.

In general, the relations $\leqq\circ$ and $\circ\leqq$ are distinct. On convex subsets of L, the relation $\leqq$ is a partial ordering. For Θ-blocks, the three relations coincide:

Lemma 7. *If A, B are Θ-blocks of L, then $A\circ\leqq B$, $A\leqq\circ B$, and $A\leqq B$ are equivalent.*

Proof. Assume that $A\leqq\circ B$, i.e., for every $a\in A$, there is a $b\in B$ with $a\leqq b$. Hence, $a=a\wedge b\in A\mathbf{\wedge}B$. Thus, $A\subseteq A\mathbf{\wedge}B$. Since $A\mathbf{\wedge}B$ is Θ-connected by Lemma 6 and A is a maximal Θ-connected subset of L, we conclude that $A=A\mathbf{\wedge}B$. Hence, if $b\in B$ is given, then $b\wedge a\in A$ for all $a\in A$. Thus, $A\circ\leqq B$. The converse is analogous.

Theorem 1 (see G. Czédli [1]). *If Θ is a tolerance on the lattice L, then L/Θ, the set of Θ-blocks, forms a lattice with respect to the ordering $\leqq$. In addition, we have $A\vee B=(A\mathbf{\vee}B)^\Theta$ and $A\wedge B=(A\mathbf{\wedge}B)_\Theta$, for all $A, B\in L/\Theta$.*

Proof. We prove that $A\vee B$ exists and equals $(A\mathbf{\vee}B)^\Theta$; the second formula follows by duality. If $C\geqq A, B$ for a Θ-block C, then trivially $C\geqq\circ A\mathbf{\vee}B$, hence $C\geqq\circ(A\mathbf{\vee}B)^\Theta$. Assuming that $(A\mathbf{\vee}B)^\Theta$ has been shown to be a Θ-block, we are finished, since then $C\geqq(A\mathbf{\vee}B)^\Theta$ and, hence, $(A\mathbf{\vee}B)^\Theta=A\vee B$.

Since $(A\mathbf{\vee}B)^\Theta$ is Θ-connected, we only have to show that $(A\mathbf{\vee}B)^\Theta$ is a maximal Θ-connected set. Let $D\supseteq(A\mathbf{\vee}B)^\Theta$ be a Θ-connected subset of L. By Lemma 6, $D\mathbf{\wedge}A$ and $D\mathbf{\wedge}B$ are Θ-connected sets. Since $A\mathbf{\vee}B\subseteq D$, we obtain $A\subseteq D\mathbf{\wedge}A$ and $B\subseteq D\mathbf{\wedge}B$ and, hence, $A=D\mathbf{\wedge}A$, $B=D\mathbf{\wedge}B$. For $d\in D$, $a\in A$, $b\in B$, we get

$d\wedge a\in A$, $d\wedge b\in B$ and $d\geqq(d\wedge a)\vee(d\wedge b)\in A\boldsymbol{\vee}B$. Now $d\Theta D$ implies that $d\Theta(A\boldsymbol{\vee}B)$, hence $d\in(A\boldsymbol{\vee}B)^{\Theta}$. Thus, $D=(A\boldsymbol{\vee}B)^{\Theta}$, as claimed.

The description of $A\vee B$ and $A\wedge B$ in Theorem 1 can be generalized.

Remark to Theorem 1. If $A_1, A_2, \ldots, A_n$ are Θ-blocks, then

$$A_1\vee A_2\vee\ldots\vee\vee A_n = (A_1\boldsymbol{\vee}A_2\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n)^{\Theta}$$

and

$$A_1\wedge A_2\wedge\ldots\wedge A_n = (A_1\boldsymbol{\wedge}A_2\boldsymbol{\wedge}\ldots\boldsymbol{\wedge}A_n)_{\Theta}.$$

Proof. By induction on n. For $n=1, 2$ we know the result. For $n\geqq 3$ we obtain

$$A_1\vee\ldots\vee A_{n-1}\vee A_n = ((A_1\vee\ldots\vee A_{n-1})\boldsymbol{\vee}A_n)^{\Theta} =$$

$$((A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_{n-1})^{\Theta}\boldsymbol{\vee}A_n)^{\Theta} \quad \text{(by the induction hypothesis)} \subseteqq (A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_{n-1}\boldsymbol{\vee}A_n)^{\Theta}.$$

The maximality of $A_1\vee\ldots\vee A_n$ implies then that $A_1\vee\ldots\vee A_n=(A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n)^{\Theta}$. The second assertion follows by duality.

We add a few observations. If $H\subseteqq L$ is a non-empty, Θ-connected subset of L, then both $(H_{\Theta})^{\Theta}$ and $(H^{\Theta})_{\Theta}$ are Θ-blocks containing H. As the next lemma shows, the first block is the smallest and the second block is the largest Θ-block containing H.

Lemma 8. *Let $H\subseteqq L$ be a non-empty, Θ-connected set.*
(1) If $D\supseteqq H$ is a Θ-connected subset of L, then $H^{\Theta}\boldsymbol{\vee}D\subseteqq H^{\Theta}$ and $H_{\Theta}\boldsymbol{\wedge}D\subseteqq H_{\Theta}$.
(2) If D is a Θ-block with $D\supseteqq H$, then $(H_{\Theta})^{\Theta}\leqq D\leqq(H^{\Theta})_{\Theta}$.

Proof. (1) Let $x\in H^{\Theta}\boldsymbol{\vee}D$, i.e., $x=z\vee d$ for some $z\in H^{\Theta}$ and $d\in D$. Then we have $x\geqq z\geqq y$ for some $y\in H$ and $x\Theta H$ (since $z\Theta H$ and $d\Theta H$ hold true). Thus, $x\in H^{\Theta}$, and so $H^{\Theta}\boldsymbol{\vee}D\subseteqq H^{\Theta}$. The proof of $H_{\Theta}\boldsymbol{\wedge}D\subseteqq H_{\Theta}$ is analogous.
(2) $D\subseteqq D\boldsymbol{\wedge}(H^{\Theta}\boldsymbol{\vee}D)\subseteqq D\boldsymbol{\wedge}H^{\Theta}\subseteqq D\boldsymbol{\wedge}(H^{\Theta})_{\Theta}$ implies that $D=D\boldsymbol{\wedge}(H^{\Theta})_{\Theta}$, i.e., $D\leqq(H^{\Theta})_{\Theta}$. $D\subseteqq D\boldsymbol{\vee}(H_{\Theta}\boldsymbol{\wedge}D)\subseteqq D\boldsymbol{\vee}H_{\Theta}\subseteqq D\boldsymbol{\vee}(H_{\Theta})^{\Theta}$ implies $D=D\boldsymbol{\vee}(H_{\Theta})^{\Theta}$, i.e., $D\geqq(H_{\Theta})^{\Theta}$.

Definition 2. If L is a lattice and Θ is a tolerance on L, then we call a Θ-connected subset H of L a *Θ-block fixing set* if there exists exactly one Θ-block D with $H\subseteqq D$.

Examples. 1. If $A_1, \ldots, A_n$ are Θ-blocks, then $A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n$ and $A_1\boldsymbol{\wedge}\ldots\boldsymbol{\wedge}A_n$ are Θ-block fixing sets.

Proof. Let $D\supseteqq A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n$ be a Θ-block, then $D\geqq A_1\vee\ldots\vee A_n=(A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n)^{\Theta}$ holds. If $d\in D$, then $d\Theta(A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n)$ and there are $a_i\in A_i$ with $d\geqq a_1\vee\ldots\vee a_n$; thus, $d\in(A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n)^{\Theta}$. We obtain $D\subseteqq(A_1\boldsymbol{\vee}\ldots\boldsymbol{\vee}A_n)^{\Theta}$ and therefore $D=A_1\vee\ldots\vee A_n$. A dual argument shows that $A_1\boldsymbol{\wedge}\ldots\boldsymbol{\wedge}A_n$ is a Θ-block fixing set.

2. If A, B, C are Θ-blocks, then $(A\vee B)\wedge C$ is not, in general, a Θ-block fixing set. The following example illustrates the claim: The 8-element-lattice L of diagram 3 has a tolerance Θ which is given by the five Θ-blocks A, B, C, D, E. Obviously, $(A\vee B)\wedge C=\{y\}$, but $\{y\}$ is not a Θ-block fixing set.

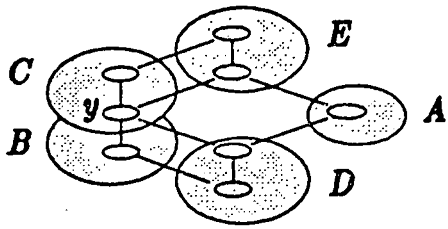

Diagram 3

3. If $H\subseteq L$ is a Θ-connected set, then H^Θ and H_Θ are Θ-block fixing sets.

Proof. If $D\supseteq H^\Theta$ is a Θ-block, then $(H^\Theta)_\Theta=((H^\Theta)_\Theta)^\Theta\leqq D\leqq((H^\Theta)^\Theta)_\Theta==(H^\Theta)_\Theta$, by Lemma 8; thus, $D=(H^\Theta)_\Theta$, and H^Θ is a Θ-block fixing set. Dually, H_Θ is a Θ-block fixing set as well.

Theorem 2. *If $H\subseteq L$ is a Θ-connected set, then the following two statements are equivalent:*

(1) *H is a Θ-block fixing set.*

(2) $(H_\Theta)^\Theta=(H^\Theta)_\Theta$.

Proof. (1) clearly implies (2), and Lemma 8 shows that (2) implies (1).

For $H\subseteq L$, let $[H]$ denote the sublattice of L generated by H.

Lemma 9. *Let Θ be a tolerance on the lattice L.*

(1) *$X, Y\subseteq L$ and $X\leqq_\circ Y\subseteq[X]$ imply $X_\Theta\subseteq Y_\Theta$.*

(2) *If X, Y are Θ-connected subsets of L with $X_\Theta\subseteq Y_\Theta$, then $(X_\Theta)^\Theta=(Y_\Theta)^\Theta$.*

Proof. (1) Choose $a\in X_\Theta$. Then $a\Theta X$ holds and there is some $x\in X$ with $a\leqq x$. We choose $y\in Y$ with $x\leqq y$, hence $a\leqq y$. $a\Theta X$ implies $a\Theta[X]$; thus, we have $a\Theta Y$. We conclude $a\in Y_\Theta$.

(2) X_Θ and Y_Θ are Θ-block fixing sets (see Example 3 preceeding Theorem 2). Because of $X_\Theta\subseteq(X_\Theta)^\Theta$ and $X_\Theta\subseteq(Y_\Theta)^\Theta$ we conclude $(X_\Theta)^\Theta=(Y_\Theta)^\Theta$.

Lemma 9 is the crux of the next theorem.

Theorem 3. *Let L be a lattice and $X=\{x_1, \ldots, x_n\}\subseteq L$, $n\in\mathbf{N}$, a Θ-connected* *finite subset of L. Then* $(X_\Theta)^\Theta = (\{x_1 \vee \dots \vee x_n\}_\Theta)^\Theta$. (In words: All finitely generated Θ-blocks are principal Θ-blocks.)

Proof. By Lemma 9, $X \equiv \circ \{x_1 \vee \dots \vee x_n\} \subseteq [X]$ implies $X_\Theta \subseteq \{x_1 \vee \dots \vee x_n\}_\Theta$; thus, $(X_\Theta)^\Theta = (\{x_1 \vee \dots \vee x_n\}_\Theta)^\Theta$.

2. Quotients of tolerances and the Second Isomorphism Theorem

Let L be a lattice, $\mathrm{Con}\, L$ the lattice of congruences on L and $\mathrm{Tol}\, L$ the lattice of tolerances on L. If Θ and Φ are tolerances on L, then we define the binary relation Θ/Φ on L/Φ as follows: $A \Theta/\Phi B$ holds if and only if there are $a \in A$, $b \in B$ with $a \Theta b$. If Θ and Φ are congruences and $\Theta \geqq \Phi$, then Θ/Φ is a congruence on L/Φ, and the well-known Second Isomorphism Theorem states that $L/\Theta \cong \cong (L/\Phi)/(\Theta/\Phi)$ holds. In general, Θ/Φ is only tolerance on L/Φ. We will show that every tolerance on an arbitrary lattice L' is of the form Θ/Φ for congruences Θ and Φ on a suitable lattice L.

Thus, let L' be a lattice and let Θ' be a tolerance relation on L'. We define the lattice L as sublattice of the direct product $L' \times (L'/\Theta')$ on the carrier set $L := \{(a, A);\ A \in L'/\Theta' \text{ and } a \in A\}$. If $\pi_1\colon L \to L'$ and $\pi_2\colon L \to L'/\Theta'$ are the restrictions of the two canonical projections from $L' \times (L'/\Theta')$ onto L' and L'/Θ', resp., then π_1 and π_2 are lattice epimorphisms. We define $\Theta := \mathrm{kernel}(\pi_2)$ and $\Phi := \mathrm{kernel}(\pi_1)$. The homomorphism theorem yields $L/\Phi \cong L'$, and the corresponding isomorphism identifies $a \in L'$ with $\pi_1^{-1}(a) \in L/\Phi$. Under this identification we get that $L' = L/\Phi$ and $\Theta' = \Theta/\Phi$.

Definition 3. Let L' be a lattice and Θ' a tolerance on L'. The lattice L and the congruences Θ, Φ just constructed are called *the lattice,* resp. *the congruences associated with* (L', Θ').

We summarize:

Theorem 4. *Let L' be a lattice, Θ' a tolerance on L'. Let L be the lattice and Θ, Φ the congruences associated with (L', Θ'). The canonical identification makes the following two statements true:*
(i) $L/\Phi = L'$, (ii) $\Theta/\Phi = \Theta'$.

In case of a congruence Θ', we get $\Phi = \omega$, $L = L'$, and $\Theta = \Theta'$ in Theorem 4. In case of a tolerance Θ' we have no natural correspondence between L/Θ and $(L/\Phi)/(\Theta/\Phi)$, but a suitable modification yields a generalized version of the Second Isomorphism Theorem. In order to derive the result, we find another way of interpreting a tolerance Θ' on a lattice L'. This interpretation associates Θ' on L' with $\Phi\circ\Theta\circ\Phi$ on L.

Lemma 10. *Let L be a lattice and Θ, Φ tolerances on L. Then the following are true:*

$$\Phi\circ\Theta\circ\Phi\in\operatorname{Tol} L,\quad \Theta\leqq\Phi\circ\Theta\circ\Phi,\quad\text{and}\quad \Phi\circ\Theta\circ\Phi/\Phi=\Theta/\Phi\in\operatorname{Tol} L/\Phi.$$

Example. Quite different from situation for congruences, not every tolerance Θ on a lattice L is of the form $\Phi\circ\Xi\circ\Phi$ for suitable tolerances Φ, Ξ. E.g., the tolerance Θ on the lattice L of Diagram 3 is not of that kind.

Theorem 5 (The Second Isomorphism Theorem). *Let L be a lattice, $\Phi\in\operatorname{Con} L$, and $\Theta\in\operatorname{Tol} L$. Then*

$$L/\Phi\circ\Theta\circ\Phi\cong(L/\Phi)/(\Theta/\Phi)=(L/\Phi)/(\Phi\circ\Theta\circ\Phi/\Phi).$$

Proof. Define $L':=L/\Phi$, $\Theta':=\Theta/\Phi$, and let $\pi\colon L\to L'$ be the natural projection. If we extend π to the respective powersets in the canonical way, then we obtain the mapping $\bar\pi\colon \operatorname{Pot}(L)\to\operatorname{Pot}(L')$ and, by restriction, $\bar\pi\colon L/\Phi\circ\Theta\circ\Phi\to\operatorname{Pot}(L')$.

(i) *Claim.* $\mathrm{A}\in L/\Phi\circ\Theta\circ\Phi$ implies that $\bar\pi(\mathrm{A})\in L'/\Theta'$.

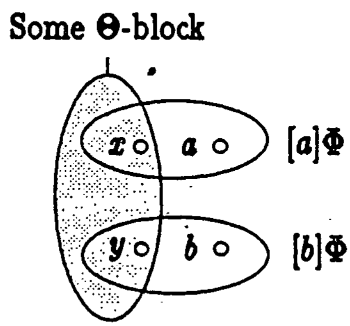

Diagram 4

Assume that $a\Phi\circ\Theta\circ\Phi b$, i.e., $a\Phi x\Theta y\Phi b$ for suitable $x\in[a]\Phi, y\in[b]\Phi$ $(a, b\in L)$. By the definition of Θ', we get $([a]\Phi)\Theta'([b]\Phi)$. Thus, $\bar\pi(\mathrm{A})$ is Θ'-connected. To show the maximality of $\bar\pi(\mathrm{A})$ with respect to Θ'-connectedness, let $[x]\Phi\in L'$ be an element with $([x]\Phi)\Theta'([a]\Phi)$ for all $a\in\mathrm{A}$. Thus, for every $a\in\mathrm{A}$, there are elements, $x'\in[x]\Phi$, $a'\in[a]\Phi$ with $x'\Theta a'$. We conclude that $x\Phi x'\Theta a'\Phi a$, i.e., $x\Phi\circ\Theta\circ\Phi a$, and, hence, $x\Phi\circ\Theta\circ\Phi\mathrm{A}$. We deduce that $x\in\mathrm{A}$ and, hence, $[x]\Phi\in\bar\pi(\mathrm{A})$. Thus, $\bar\pi(\mathrm{A})$ is a Θ'-block.

(ii) *Claim.* $A\in L'/\Theta'$ implies that $\bar\pi^{-1}(A)\in L/\Phi\circ\Theta\circ\Phi$.

Let $\mathrm{A}:=\bar\pi^{-1}(A)$. Clearly, $\bar\pi(\mathrm{A})=A$. Choose $a, b\in\mathrm{A}$ arbitrarily. Then $[a]\Phi, [b]\Phi\in A$ implies that $([a]\Phi)\Theta'([b]\Phi)$; thus, $a\Phi x\Theta y\Phi b$ holds for suitable

$x \in [a]\Phi$, $y \in [b]\Phi$, i.e., we have $a\Phi \circ \Theta \circ \Phi b$. Thus, $\mathbf{A}$ is $\Phi \circ \Theta \circ \Phi$-connected. We embed $\mathbf{A}$ in some $\Phi \circ \Theta \circ \Phi$-block $\mathbf{A}^*$ and obtain $\bar{\pi}(\mathbf{A}) = A \subseteqq \bar{\pi}(\mathbf{A}^*)$. Since A and, by (i), $\bar{\pi}(\mathbf{A}^*)$ are Θ'-blocks, we obtain $A = \bar{\pi}(\mathbf{A}^*)$ and, hence, $\mathbf{A}^* \subseteqq \bar{\pi}^{-1}(A) = \mathbf{A}$. Thus, $\mathbf{A} = \mathbf{A}^*$.

(i), (ii) and the surjectivity of π immediately imply (iii) and (iv) below.

(iii) $\bar{\pi}^{-1}(\bar{\pi}(\mathbf{A})) = \mathbf{A}$ holds for all $\Phi \circ \Theta \circ \Phi$-blocks $\mathbf{A}$ of L.

(iv) $\bar{\pi}(\bar{\pi}^{-1}(A)) = A$ holds for all Θ'-blocks A of L'.

(v) Statements (i) to (iv) prove that the restriction $\bar{\pi}\colon L/\Phi \circ \Theta \circ \Phi \to L'/\Theta'$ is a bijection. Finally, we show that $\bar{\pi}$ is even a lattice-homomorphism: $\bar{\pi}^{-1}(A) = \{a \in L;\ [a]\Phi \in A\}$ holds for every $A \in L'/\Theta'$. If, therefore, $A, B \in L'/\Theta'$ are arbitrarily chosen, then we get:

$$\bar{\pi}^{-1}(A \vee B) = \{x \in L;\ [x]\Phi \in A \vee B\} \supseteqq$$
$$\supseteqq \{a \vee b;\ a, b \in L \text{ and } [a]\Phi \in A,\ [b]\Phi \in B\} = \bar{\pi}^{-1}(A) \boldsymbol{\vee} \bar{\pi}^{-1}(B).$$

The last set is a $\Phi \circ \Theta \circ \Phi$-block fixing set. Thus, there is exactly one $\Phi \circ \Theta \circ \Phi$-block of L containing $\bar{\pi}^{-1}(A) \boldsymbol{\vee} \bar{\pi}^{-1}(B)$, namely $\bar{\pi}^{-1}(A) \vee \bar{\pi}^{-1}(B)$. Thus, we proved that $\bar{\pi}^{-1}(A \vee B) = \bar{\pi}^{-1}(A) \vee \bar{\pi}^{-1}(B)$ holds. Similarly, $\bar{\pi}^{-1}(A \wedge B) = \bar{\pi}^{-1}(A) \wedge \bar{\pi}^{-1}(B)$ holds.

3. Products of lattice varieties

If $\mathbf{V}$ and $\mathbf{W}$ are two varieties of lattices, then the product $\mathbf{V} \circ \mathbf{W}$ consists of all lattices L for which there is some congruence Θ satisfying the following two properties:

(i) All Θ-blocks of L are in $\mathbf{V}$,

(ii) $L/\Theta \in \mathbf{W}$.

We combine these two conditions by saying that Θ *establishes that L is in* $\mathbf{V} \circ \mathbf{W}$.

G. Grätzer and D. Kelly [6] give an overview of these variety products. $\mathbf{V} \circ \mathbf{W}$ is not, in general, a variety. However, one knows that the variety generated by $\mathbf{V} \circ \mathbf{W}$ is $\boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$, the class of all holomorphic images of lattices in $\mathbf{V} \circ \mathbf{W}$. R. N. McKenzie conjectured that the variety $\boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$ can be characterized as follows: "A lattice L' is contained in $\boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$ if and only if there is a tolerance Θ' on L' such that all Θ'-blocks of L' are in $\mathbf{V}$ and $L'/\Theta' \in \mathbf{W}$." The next theorem answers one direction of the conjecture in the affirmative.

Theorem 6. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be varieties of lattices. Let L' be a lattice with a tolerance Θ' that satisfies the following two properties:*

(i) *All Θ'-blocks are in* $\mathbf{V}$,

(ii) $L'/\Theta' \in \mathbf{W}$.

Then $L' \in \boldsymbol{H}(\mathbf{V} \circ \mathbf{W})$.

Proof. Let L be the lattice and let Θ, Φ be the congruences associated with (L', Θ'). Of course, $\Theta \cap \Phi = \omega$ and $L/\Theta \cong L'/\Theta'$. Thus, $L/\Theta \in \mathbf{W}$. If $\pi_1: L \to L'$ and $\pi_2: L \to L'/\Theta'$ are the projections yielding Φ and Θ, then the Θ-blocks of L are of the form $\pi_2^{-1}(A_0)$ for fixed $A_0 \in L'/\Theta'$. $\pi_2^{-1}(A_0) = \{(a, A_0);\ a \in A_0\} \cong A_0$ shows that $\pi_2^{-1}(A_0) \in \mathbf{V}$ holds. Thus, Θ establishes that $L \in \mathbf{V} \circ \mathbf{W}$. The projection π_1 yields $L' \in \mathbf{H}(L) \subseteq \mathbf{H}(\mathbf{V} \circ \mathbf{W})$.

In order to tackle the opposite direction of the above conjecture, we begin with some fixed lattice $L' \in \mathbf{H}(\mathbf{V} \circ \mathbf{W})$. $L' \in \mathbf{H}(\mathbf{V} \circ \mathbf{W})$ means $L' \cong L/\Phi$ for some $L \in \mathbf{V} \circ \mathbf{W}$ and a suitable congruence Φ on L. $L \in \mathbf{V} \circ \mathbf{W}$ is established by some congruence Θ on L. Then $\Theta' := \Theta/\Phi$ is a tolerance on L', and McKenzie's conjecture seems to be based on the hope that (i) all Θ'-blocks of L' are in $\mathbf{V}$ and (ii) $L'/\Theta' \in \mathbf{W}$ is always true. The next and last theorem states that the first assertion is valid. An example will show that the second one is, in general, not true. This suggests that the answer to McKenzie's conjecture is in the negative.

Theorem 7. *Let* $\mathbf{V}$, $\mathbf{W}$ *be lattice varieties and assume that* $L' \in \mathbf{H}(\mathbf{V} \circ \mathbf{W})$. *Then* $L' \cong L/\Phi$ *for some* $L \in \mathbf{V} \circ \mathbf{W}$ *and some congruence* Φ *on* L. *Let* Θ *be a congruence on* L *establishing* $L \in \mathbf{V} \circ \mathbf{W}$. *If* $\Theta' := \Theta/\Phi$, *then all* Θ'*-blocks of* L' *are in* $\mathbf{V}$.

Proof. We will show that for every finite Θ'-connected set $\{[a_0]\Phi, \ldots, [a_n]\Phi\}$ of Φ-blocks there is a Θ-connected subset $\{u_0, \ldots, u_n\}$ of L with $u_i \in [a_i]\Phi$. This suffices, since then every Θ'-block satisfies every identity which is satisfied by every Θ-block. Since tolerance blocks are sublattices, we may assume that $[a_0]\Phi < [a_i]\Phi$ holds for all $i = 1, 2, \ldots, n$. By the definition of Θ', we find suitable $b_i \in [a_i]\Phi$ and $a_0^i \in [a_0]\Phi$ with $a_0^i < b_i$ and $a_0^i \Theta b_i$ $(i = 1, 2, \ldots, n)$. Let $a := a_0^1 \vee a_0^2 \vee \ldots \vee a_0^n \in [a_0]\Phi$. Due to $b_i \Theta a_0^i$ and $a_0^i \Theta a_0^i$, we get $(b_i \vee a)\Theta a$ and $b_i \vee a \in [a_i]\Phi$ for $i = 1, 2, \ldots, n$. With $u_0 := a$ and $u_i := b_i \vee a$, $i = 1, 2, \ldots, n$, our claim has been proved.

Example. We modify an example of G. Czédli [1] to show that, under the hypotheses of Theorem 7, we cannot, in general, conclude that $L'/\Theta' \in \mathbf{W}$. To do so, we describe a distributive lattice L and two congruences Θ, Φ on L (with $\Theta \cap \Phi = \omega$) such that $L/\Phi \circ \Theta \circ \Phi$ $(\cong (L/\Phi)/(\Theta/\Phi)$, by Theorem 5) is not distributive. Let L_5 be the 5-element lattice on $\{1, 2, 3, 4, 5\}$ with $1 < 2 < 3 < 4 < 5$. Then $L := (L_5 \times L_5) \setminus \{(4, 1), (5, 1)\} \in \mathbf{D}$ (variety of distributive lattices). On L_5 we define the congruences Φ_1, Φ_2, Θ_1, Θ_2 via the corresponding congruence blocks listed below:

$$\Phi_1: \{\{1, 2\}, \{3, 4\}, \{5\}\}$$
$$\Phi_2: \{\{1\}, \{2, 3\}, \{4\}, \{5\}\}$$
$$\Theta_1: \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$$
$$\Theta_2: \{\{1, 2\}, \{3\}, \{4, 5\}\}.$$

Then $\Phi_1 \times \Phi_2$, $\Theta_1 \times \Theta_2 \in \mathrm{Con}\,(L_5 \times L_5)$, and we define $\Phi := \Phi_1 \times \Phi_2|_L$, $\Theta := \Theta_1 \times \Theta_2|_L$. Diagram 5 shows the Θ-blocks (indicated by bold border lines) and the Φ-blocks (indicated by normal border lines) on L.

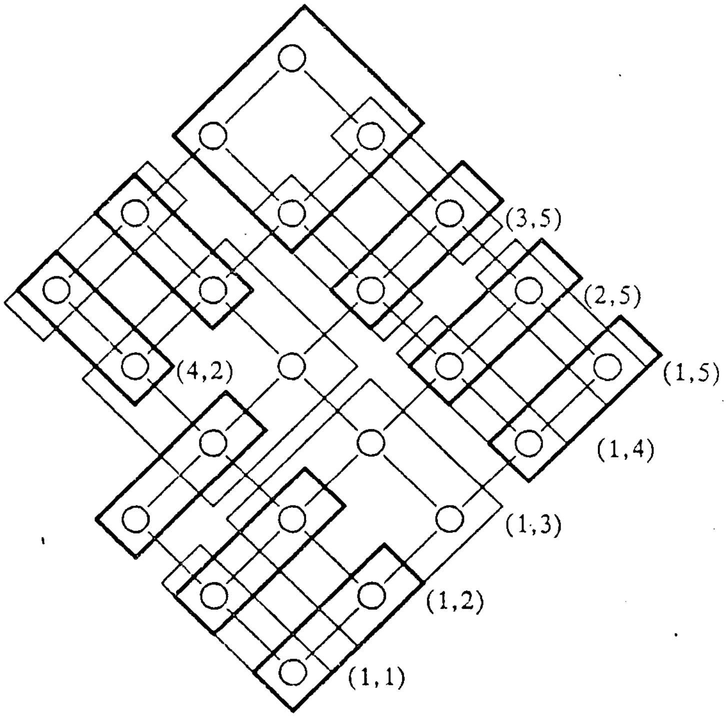

Diagram 5

Diagram 6 shows the $\Phi \circ \Theta \circ \Phi$-blocks on L. We recognize that $L/\Phi \circ \Theta \circ \Phi \cong$ $\cong N_5 \notin \mathbf{D}$.

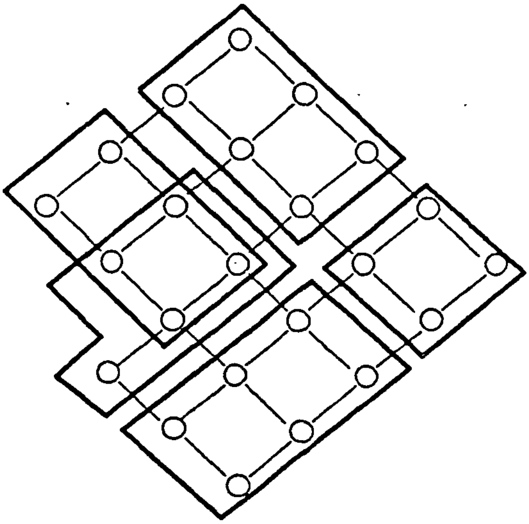

Diagram 6

Note. The conjecture referred to in this paper has in the meanwhile been answered in the negative by E. FRIED and G. GRÄTZER [2].

Note added in proof (September 19, 1990) by G. Grätzer. The second author, my former student, friend, and colleague, tragically died last year. I shall miss him.

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A characterization of o-distributive semilattices

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The notion of a distributive ordered set which generalizes the notion of a distributive lattice is introduced in [3], where there are shown some properties of such ordered sets. In [2] there are described ordered sets having a similar importance for distributive ordered sets as the pentagon and the diamond have for distributive lattices, i.e. on certain conditions they are not included in a distributive ordered set (e.g. as its strong subset) and each non-distributive ordered set contains at least one of those sets as an LU-subset. (For the definitions of an LU-subset and a strong subset see below.)

The aim of this paper is to describe the semilattices which are distributive ordered sets.

Let $A=(A, \leqq)$ be an ordered set. If $B \subseteq A$, then we denote

$$L_A(B) = \{x \in A;\ x \leqq b, \text{ for all } b \in B\},$$

$$U_A(B) = \{y \in A;\ y \geqq b, \text{ for all } b \in B\}.$$

If it is not a danger of misunderstanding, we write also $L(B)$ and $U(B)$ instead of $L_A(B)$ and $U_A(B)$. For $B=\{a_1, \ldots, a_n\}$ we use also the forms $L(B)=L(a_1, \ldots, a_n)$ and $U(B)=U(a_1, \ldots, a_n)$.

Definition 1. An ordered set A is called *distributive* if

$$L(U(L(a, c), L(b, c))) = L(U(a, b), c) \quad \text{for all } a, b, c \in A.$$

Remark 1. It is clear that in any ordered set A it holds $L(U(L(a, c), L(b, c))) \subseteq L(U(a, b), c)$ for all $a, b, c \in A$. Hence for the distributivity of an ordered set it suffices to verify only the identity with the opposite inclusion.

Remark 2. A lattice A is distributive if and only if it is a distributive ordered set. (See [3].)

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Recall that a semilattice $A=(A, \leqq, \vee)$ is called distributive (see [1, p. 135]) if for any $a, b, x \in A$ it holds the following condition:

If $x \leqq a \vee b$, then there exist $a_1, b_1 \in A$, $a_1 \leqq a$, $b_1 \leqq b$ such that $x = a_1 \vee b_1$.

To distinguish two notions of distributivity, a semilattice which is simultaneously a distributive ordered set will be called an o-distributive semilattice.

We will show a connection between these notions.

Proposition 1. *Every distributive semilattice is o-distributive.*

Proof. If $A=(A, \vee)$ is a semilattice, $a, b, c \in A$, then $L(U(a, b), c) = L(a \vee b, c)$. Let A be a distributive semilattice, $a, b, c, x \in A$, $x \leqq c$, $x \leqq a \vee b$. Then there exist $a_1, b_1 \in A$, $a_1 \leqq a$, $b_1 \leqq b$ such that $x = a_1 \vee b_1$. Let $y \in U(L(a, c), L(b, c))$. Then $a_1 \leqq y$, $b_1 \leqq y$, hence $x = a_1 \vee b_1 \leqq y$, and therefore $L(a \vee b, c) \subseteq L(U(L(a, c), L(b, c)))$.

Remark 3. The converse implication is not true. For example, the semilattice $A = \{a, b, c\}$, where $a < c$, $b < c$ (see Fig. 1), is o-distributive but it is not distributive.

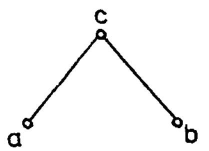

Fig. 1

Definition 2. a) A subset M of an ordered set A is said to be an *LU-subset* of A, if for each $a, b \in M$:

(i) $L_M(a, b) = \emptyset$ if and only if $L_A(a, b) = \emptyset$;

(ii) $U_M(a, b) = \emptyset$ if and only if $U_A(a, b) = \emptyset$.

b) A subsemilattice M of a semilattice $A=(A, \vee)$ which is an LU-subset of A (i.e. M satisfies the condition (i)) is called an LU-subsemilattice of A.

Theorem 2. *Let a semilattice $A=(A, \vee)$ do not be o-distributive. Then it contains an LU-subsemilattice isomorphic to one of the ordered sets M_2, M_4, N_3, N_4.* (See Fig. 2.)

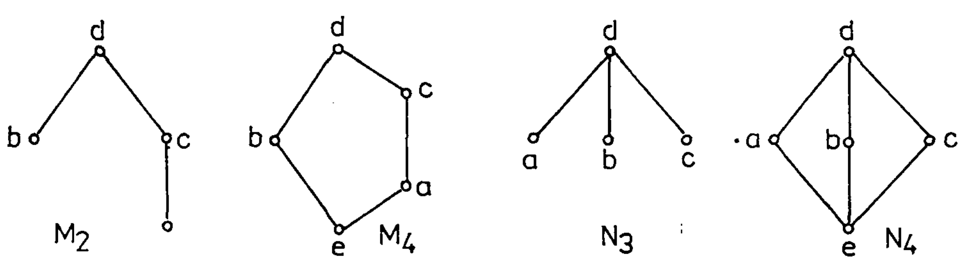

Fig. 2

Proof. If a semilattice A is not o-distributive, then there exist $a, b, c \in A$ such that

$$L(U(L(a,c),\ L(b,c))) \subset L(a \vee b, c).$$

I. Let $a<c$. Then $L(U(L(a,c),\ L(b,c))) = L(U(a, L(b,c)))$, and thus $L(U(a, L(b,c))) \subset L(a \vee b, c)$. Clearly $a \| b$, $b \| c$.

(a) Firstly let us suppose $L(b,c)=\varnothing$. Then there exists $x \in L(a \vee b, c)$ such that $x \not\leqq a$.

(α) Let $x>a$. Then $a \vee b = b \vee x$, $a \vee b > b$, $b \| x$. From that we also have $a \vee b > x$. Therefore the set $T_1 = \{a, b, x, a \vee b\}$ is a subsemilattice of A. Furthermore $L(a,b) \subseteq L(b,x) \subseteq L(b,c) = \varnothing$, hence T_1 is an LU-subsemilattice of A isomorphic to M_2.

(β) Let $x \| a$. Let us denote $T_2 = \{a, b, a \vee x, a \vee b\}$. We have $a \vee x \leqq a \vee b$ and $a < a \vee x$. Furthermore $a \vee b \not\leqq c$. In the case $c < a \vee b$, we obtain $a \vee b \geqq a \vee x$, in the case $c \| a \vee b$, we have $a \vee x < c$, $a \vee x < a \vee b$. Therefore it always holds $a \vee x < a \vee b$. In addition, we have $b < a \vee b$. Let us show that $b \| a \vee x$. In fact, if $a \vee x \leqq b$, then $a<b$, a contradiction, and if $b < a \vee x$, then $a \vee b \leqq a \vee x$, a contradiction, too.

Therefore T_2 is a subsemilattice of A, and because $L(a,b) \subseteq L(b, a \vee x) \subseteq L(b,c) = \varnothing$, T_2 is an LU-subsemilattice of A isomorphic to M_2.

(b) Let now $L(b,c) \neq \varnothing$ and let $v \in L(b,c)$. Since $L(U(a, L(b,c))) \subset L(a \vee b, c)$, there exist $x \in L(a \vee b, c)$, $y \in U(a, L(b,c))$ such that $x \not\leqq y$.

(α) Let $x>y$. Let us denote $T_3 = \{b, x, y, v, a \vee b\}$. Then from $a<x$ we obtain $a \vee b \leqq x \vee b$, and since evidently $x \vee b \leqq a \vee b$, we have $y \vee b = a \vee b$. Further it is clear that $v<b$ and $v<y$. Since $c \| b$, we have $x < a \vee b$. If $b \geqq x$, then $b>a$, and if $b \leqq x$, then $x = a \vee b$, hence it must hold $b \| x$. Analogously we can prove $b \| y$. But this means that T_3 is an LU-subsemilattice of A isomorphic to M_4.

(β) Let $x \| y$. Let us denote $T_4 = \{b, a \vee v, x \vee a \vee v, v, a \vee b\}$. Since $v<b$, $x \leqq a \vee b$ and $a<b$, we have $x \vee a \vee v \leqq a \vee b$. Let us suppose $x \vee a \vee v = a \vee b$. Then $x \vee a \vee v > b$, hence $c \vee x \vee a \vee v \geqq b \vee c$. But $c \vee x \vee a \vee v = c$, therefore $c \geqq b$, a contradiction. Thus it must be $x \vee a \vee v < a \vee b$.

Since $x \| y$, we obtain $x \not\leqq a \vee v$, hence $x \vee a \vee v \neq a \vee v$, and so $a \vee v < x \vee a \vee v$. Further it is evident that $v < a \vee v$, $v<b$, $b < a \vee b$. At the same time, if $b \geqq a \vee v$, then $b \geqq a$, and if $b \leqq a \vee v$, then $b \leqq c$, a contradiction. Thus $b \| a \vee v$. Similarly $x \vee a \vee v \| b$.

Therefore T_4 is an LU-subsemilattice of A isomorphic to M_4.

II. Now, we shall observe the case $a \| c$. It is evident that then $a \| b$ and $c \not\leqq b$. We can suppose $b \| c$, otherwise we would obtain the same results as for the case I.

(a) First let us suppose $a \vee b < a \vee b \vee c$, $a \vee c < a \vee b \vee c$, $b \vee c < a \vee b \vee c$.

(α) Let $L(a,b) = L(a,c) = L(b,c) = \varnothing$. Then $L(U(L(a,c),\ L(b,c))) = \varnothing$, but

$L(a\vee b, c)\neq\emptyset$. Let $x\in L(a\vee b, c)$. Then $R_1=\{x, a\vee b, a\vee c, b\vee c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

(β) If e.g. $L(a, b)\neq\emptyset$, $d\in L(a, b)$, then $R_2=\{d, a\vee b, a\vee c, b\vee c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

(b) Let $a\vee b=a\vee b\vee c$, $a\vee c<a\vee b$, $b\vee c<a\vee b$.

(α) Let $L(a, b)=L(a, c)=L(b, c)=\emptyset$. If $L(a\vee c, b)=\emptyset$, then $R_3=\{a, b, a\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_2.

If $L(a\vee c, b)\neq\emptyset$, $d\in L(a\vee c, b)$, then $R_4=\{d, b, a\vee c, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_4.

(β) If $L(a, b)\neq\emptyset$, $e\in L(a, b)$, then $R_5=\{e, b, a\vee c, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_4.

(γ) If e.g. $L(a, c)\neq\emptyset$, $f\in L(a, c)$, then $R_6=\{f, a, a\vee c, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_4.

(c) Let us suppose $a\vee b=a\vee c=a\vee b\vee c$, $b\vee c<a\vee b$.

(α) Let $L(a, b)=L(a, c)=L(b, c)=\emptyset$. If $L(a, b\vee c)=\emptyset$, then $R_7=\{a, b, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_2.

Let $L(a, b\vee c)\neq\emptyset$, $g\in L(a, b\vee c)$. Then $L(b, g)=L(c, g)=\emptyset$. If $b\vee g=c\vee g=b\vee c$, then $R_8=\{b, g, c, b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_3. If $b\vee g<b\vee c$, then $R_9=\{g, b\vee g, b\vee c, a\vee b, a\}$ is an LU-subsemilattice of A isomorphic to M_4.

(β) Let $L(a, b)\neq\emptyset$, $h\in L(a, b)$. Then $R_{10}=\{h, b, b\vee c, a\vee b, a\}$ is an LU-subsemilattice of A isomorphic to M_4. (Similarly for $L(a, c)\neq\emptyset$.)

(γ) Let $L(a, b)=L(a, c)=\emptyset$, $L(b, c)\neq\emptyset$. If $L(a, b\vee c)=\emptyset$, then R_7 is an LU-subsemilattice of A. Suppose $L(a, b\vee c)\neq\emptyset$, $g\in L(b, c)$, $h\in L(a, b\vee c)$. We have $h\vee g\not\leqq b$, $h\vee g\not\leqq c$, $h\vee g\leqq b\vee c$. Let $b<h\vee g$. If $h\vee g<b\vee c$, then $R_{11}=\{h, h\vee g, b\vee c, a\vee b, a\}$ is an LU subsemilattice of A isomorphic to M_4. If $h\vee g=b\vee c$, then $R_{12}=\{g, h, b, b\vee c\}$ is an LU-subsemilattice of A isomorphic to M_2. (For $c<h\vee g$, we can prove similarly.)

Let $b\| h\vee g$, $c\| h\vee g$. If $b\vee h\vee g=b\vee c$ and $c\vee h\vee g=b\vee c$, then $R_{13}=\{h, b, h\vee g, c, b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4. If $b\vee h\vee g<b\vee c$ or $c\vee h\vee g<b\vee c$, respectively, then $R_{14}=\{h, c, b, b\vee h\vee g, b\vee c\}$ or $R_{15}=\{h, b, c, c\vee h\vee g, b\vee c\}$, respectively, is an LU-subsemilattice of A isomorphic to M_4.

(d) The case $a\vee c=b\vee c=a\vee b\vee c$, $a\vee b<a\vee c$ can be proved analogously as the case (c).

(e) Let us suppose $a\vee b=a\vee c=b\vee c=a\vee b\vee c$.

(α) If $L(a, b)=L(a, c)=L(b, c)=\emptyset$, then $R_{16}=\{a, b, c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_3.

(β) Let e.g. $L(a, b)\neq\emptyset$, $d\in L(a, b)$. If $d\vee c<a\vee b\vee c$, then $R_{17}=\{d, a, b, d\vee c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

Let $d \vee c = a \vee b \vee c$ and let $L(b, c) = \emptyset$ or $L(a, c) = \emptyset$, respectively. Then $R_{18} = \{d, b, c, a \vee b \vee c\}$ or $R'_{18} = \{d, a, c, a \vee b \vee c\}$, respectively, is an LU-subsemilattice of A isomorphic to M_2.

Finally, let us observe the case $L(a, b) \neq \emptyset$, $L(a, c) \neq \emptyset$, $L(b, c) \neq \emptyset$. Let $d \in L(a, b)$, $e \in L(a, c)$, $f \in L(b, c)$. If e.g. $L(e, f) \neq \emptyset$, $g \in L(e, f)$, then $R_{19} = \{g, a, b, c, a \vee b \vee c\}$ is an LU-subsemilattice of A isomorphic to N_4. Hence, let $L(d, e) = L(d, f) = L(e, f) = \emptyset$. Since $L(a \vee b, c) = L(c)$, it exists (by the assumption) an element $x \in U(L(a, c), L(b, c))$ such that $c \nleqq x$. For x we have $x \geqq e$, $x \geqq f$, thus it must be $c > e \vee f$. If now $a \vee f > c$, then $R_{20} = \{e, a, e \vee f, c, a \vee f\}$ is an LU-subsemilattice of A isomorphic to M_4.

Let $a \vee f \parallel c$. If $a \vee f > a$, then $R_{21} = \{e, a, a \vee f, c, a \vee b \vee c\}$ is an LU-subsemilattice of A isomorphic to M_4. If $a \vee f = a$, then $R_{22} = \{f, a, b, c, a \vee b \vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

All remaining possibilities of the connections among a, b, c would lead to some variants of the preceding cases only.

Remark 4. In [2] it is proved for any ordered set A that if A is non-distributive, then it contains an LU-subset isomorphic to some of ordered sets M_1, M_2, M_3, M_4, M_5, M_6, N_1, N_2, N_3, N_4, N_5. (See Fig. 2 and 3.)

But for the case of semilattices, the constructions of respective LU-subsets from [2] do not lead to subsemilattices.

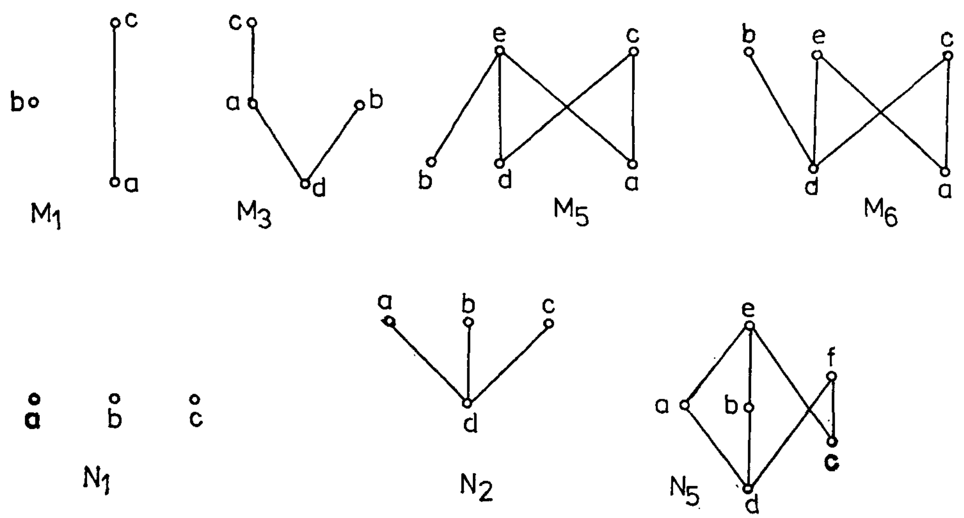

Fig. 3

Definition 3. A subset M of an ordered set A is called *strong* if for any $a, b \in M$ it holds:

(i) $L_A(U_M(a, b)) = L_A(U_A(a, b))$;

(ii) $U_A(L_M(a, b)) = U_A(L_A(a, b))$.

In [2] it is shown that if M is a strong subset of A such that $U_A(a, b) \neq \{1\}$ and $L_A(a, b) \neq \{0\}$ (where 1 or 0 denotes the greatest or the least element of A, respectively — if they exist), then M is an LU-subset of A. Furthermore, any strong subset of an ordered set A which is a semilattice with respect to the induced order, is a subsemilattice of A.

Therefore, the following theorem is similar to the converse of Theorem 2.

Theorem 3. *If a semilattice $A = (A, \vee)$ contains an LU-subsemilattice isomorphic to M_2 or to N_3, respectively, or if it contains a strong subsemilattice isomorphic to M_4 or to N_4, respectively, then A is non-o-distributive (and so non-distributive, too).*

Proof. The assertion follows from [2, Theorems 4 and 7]. It is clear that the non-distributivity of A for the cases of the strong subsemilattices M_2 and M_3 also directly follows from the fact that A is not (in those cases) lower directed.

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Non type-preserving automorphism groups of buildings and normalizing Tits systems

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0. Introduction

It is known that a very important class of groups, including real and complex reductive Lie groups and algebraic groups [1], [4], [7], [11], [12], [13], finite simple groups other than the alternating or sporadic groups [12], and also some infinite dimensional groups associated to Kac—Moody Lie algebras [15], [16] and some infinite dimensional transformation groups [9], give rise to a Tits system (or BN pair) [2], [12] and hence act on some simplicial complex which satisfies very striking geometric and combinatorial properties, axiomatized in the theory of buildings [2], [12].

A basic property of a building (for definitions, see Sec. 1) is that it admits "type mappings", the set of which is parametrized by the group of permutations of the vertices of a given chamber [12]. Furthermore, a group with Tits system acts on its building in a type preserving way [12]. By a Theorem of J. Tits ([12], 3.11, p. 44), there is a certain converse to this situation: any group of type preserving automorphisms of the building which acts transitively on the set of pairs (C, A), where C is a chamber of given type and A an apartment containing it, is a group with Tits system, with respect to B and N the stabilizers of C and A, respectively.

In this paper we consider groups of non-type preserving automorphisms of a building which satisfy the analogous condition for chambers and apartments whose type is not fixed. Non type-preserving elements of such groups (the "polarities") and their centralizers play an important role in the theory of simple groups (see for instance [14]). We show that such a group together with B and N as above, gives rise to a "normalizing Tits system", a notion which we have introduced in [9]. This means that G satisfies all the hypotheses of a Tits system except for the property of groups with Tits system that all elements of any generating set S of the Weyl

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group fail to normalize B. Under our hypotheses, the subgroup of G preserving the type still acts transitively on the set of pairs of oriented chambers and apartments, as above and hence, by Tits' Theorem, gives rise to a Tits system. We use this fact in the proof that G has a *normalizing* Tits system. The subgroup of type preserving automorphisms is a normal subgroup of G. When the building is of finite rank, then it is of finite index in G. In [9] we had given these results in the special case where the building is a homogeneous tree.

1. Basic definitions

In this paragraph we establish some basic notation and briefly recall a few definitions and results concerning buildings, taken from [12].

Given any set X, we denote the group of all bijective maps of X by $S(X)$. By a simplicial complex I we shall here mean a set $M(I)$ together with a family of subsets $\mathrm{Sim}\,(I)$, the simplices of I, which is closed under taking subsets. Given a simplex S, we denote the set of points of $I(S)$ by $M(S)$. I is completely determined by the set of maximal elements of $\mathrm{Sim}\,(I)$, which we shall denote by $\mathrm{Ch}\,(I)$. We shall write the action of the automorphism group of I on $M(I)$ or $\mathrm{Sim}\,(I)$ on the right, $m \to m \cdot g$, except when we wish to name the homomorphism α: $\mathrm{Aut}\,(I) \to S(M(I))$ or $S(\mathrm{Sim}\,(I))$, in which case we shall write $\alpha(g)(m)$, etc.

Definition 1.1. A simplicial complex $A=(M(A), \mathrm{Sim}\,(A))$ is said to be "*thin*" iff the following conditions are verified:

(i) For any maximal simplices, C, C' there exists a chain of maximal simplices, γ: $C=C(0), \ldots, C(n)=C'$ such that the successive intersections $C(i) \cap C(i+1)$ are of codimension 1 in both $C(i)$ and $C(i+1)$. A maximal simplex is called a *chamber*. Two chambers whose intersection is of codimension 1 are called *adjacent*, an their intersection is called their wall. γ is called a *gallery* between C and C'.

(ii) If (C, C') and (D, D') are two pairs of adjacent chambers, such that $C \cap C' = D \cap D'$ then the sets $\{C, C'\}$ and $\{D, D'\}$ coincide.

One of the basic results about thin complexes is the following result on the set of chambers: Given a set X, a bijection from some ordinal o onto X is called a *tuple* in X. By fixing a given tuple, we can identify the set of all tuples in X with $S(X)$.

Definition 1.2. Given a simplicial complex I, and a simplex X of I, we shall denote the set of ordered tuples in $M(X)$ by $\langle X \rangle$; we can identify $\langle X \rangle$ with $S(M(X))$. We denote by $\langle \mathrm{Ch}\,(I) \rangle$ the set $\mathrm{Ch}\,(I) \times \mathrm{Typ}\,(I)$ (see Definitions 1.3. and 1.4.).

Definition 1.3. A simplicial complex $I=(M(I), \mathrm{Sim}\,(I))$ together with a family of subcomplexes Ap, called *apartments*, is called a *building* iff the following conditions are satisfied:

i) Any subcomplex A in Ap is thin.

ii) For any two chambers C, C' of I, there exists an apartment A which contains both C and C'.

iii) If the intersection of two apartments A, A' contains two chambers C and D, then there exists an isomorphism $q\colon A \to A'$ which fixes C and D, as well as all their faces.

For a chamber C in a building I, we shall let $\mathrm{Sim}(C)$ and $M(C)$ denote the simplices and points of I contained in C, respectively.

Definition 1.4. A *type mapping* (relative to a fixed chamber C) for a building is a mapping $t\colon M(I) \to M(C)$ which restricts to a bijection on $M(C')$ for each chamber C'. We shall denote the set of type mappings by $\mathrm{Typ}(I)$.

The basic property of type mappings of a building is the following result ([12]' Ch. 3):

Theorem 1.1. *Any type mapping t is uniquely determined by the permutation $\sigma\colon M(C) \to M(C)$ which t induces by restriction to $M(C)$.*

One has an obvious action of $S(M(C))$ on the set of type mappings of I, given by the formula

$$\sigma(t)(m) = \sigma(t(m)). \tag{1}$$

By Theorem 1.1, this action is regular (i.e. free and transitive), and hence we can identify a type mapping with a permutation of $M(C)$. Indeed, we shall identify the permutation σ with the unique type mapping inducing σ on $M(C)$. Given a point m of I and a type map t, $t(m)$ is called the type of m, and given a wall S between two chambers, the cotype of S is the unique element of $M(C)$ which is not in the image $t(S)$.

The notion of a Tits system (or BN pair) was first defined in [11]. For some of its implications see also [1], [2], [3], [7], [12], [13], as well as the papers referred to in [12].

Definition 1.5. Let G be a group, N and B two subgroups, S a set. We say that a quadruple (G, B, N, S) is a Tits system if and only if

(T. 1) the subgroups B and N generate G.

(T. 2) $B \cap N$ is normal in N.

Furthermore the group $W = N/B\cap N$ has a set S of generators of order 2 such that

(T. 3) for each w in W, s in S, the inclusion $B\cdot w\cdot B\cdot s \subseteq B\cdot w\cdot s\cdot B \cup B\cdot w\cdot B$ holds

(T. 4) for each s in S, $s\cdot B\cdot s \neq B$.

Clearly, in (T.3) one can also write $B \cdot w \cdot B \cdot s \cdot B$ instead of $B \cdot w \cdot B \cdot s$. The principal result of this paper is a generalization of the following result of J. TITS [12, 3.11, p. 46].

Theorem 1.2. *Let G be a group of type-preserving automorphisms of a building acting transitively on the set of pairs (C, A), where C is a chamber (of fixed type) and A an apartment containing it. Let B and N be the stabilizers in G of C and of A, respectively. Then $B \cap N$ is normal in N, and there exists a set of generators S in $W = N/N \cap B$ such that the quadruple (G, B, N, S) forms a Tits system.*

2. Non-type preserving automorphisms

We shall fix a building I. We consider the action of $G = \mathrm{Aut}(I)$ and certain subgroups of it (which we shall define later) on $\langle \mathrm{CH}(I) \rangle$ (Definition 1.2). G will always denote a subgroup of $\mathrm{Aut}(I)$. We consider throughout this paper a fixed chamber C; and t the unique type map t, defined with respect to C which induces the identity on $M(C)$. By means of it, we can identify $\langle \mathrm{CH}(I) \rangle$ with $\mathrm{Ch}(I) \times S(M(C))$. By this identification a typed chamber $\langle D \rangle$ for which $t(\langle D \rangle) = \mathrm{id}$ is identified with the pair (D, id). The reader is warned that this contrasts with the usual terminology of buildings ([12] for instance), where the letters C, D, etc. refer to chambers with fixed type under a given type mapping, whereas here we use C, D to denote the "abstract" chambers. Similar remarks apply to apartments and simplices of a building. When we wish to emphasize this point, we shall speak of abstract chambers, apartments, etc.

G acts on $\mathrm{Ch}(I)$ in the obvious way. We shall denote this action by $p\colon G \to S(\mathrm{Ch}(I))$ and shall sometimes abbreviate $p(g)(D)$ by $D \cdot g$. G acts also on $\mathrm{Typ}(I)$:

$$q(g)(a)(m) = a(m \cdot g), \tag{2}$$

for any type map a and any m in $M(I)$.

As an immediate consequence of Theorem 1.1, one has

Lemma 2.1. *The stabilizer $G(a)$ in G of any type mapping a is normal in G. $G/G(a)$ acts freely on* $\mathrm{Typ}(I)$. *We denote the G-orbit of t by* $\mathrm{Typ}(G)$. *It forms a subgroup of $S(M(C))$.*

Definition 2.1. An automorphisms g is called *type preserving* if and only if it lies in $G(a)$, for some (and hence by the Lemma for all) type mappings. We denote by $G(0)$ the subgroup of type preserving automorphisms in G.

Finally, there is an action of G on $\langle \mathrm{Ch}(I)\rangle = \mathrm{Ch}(I) \times S(M(C))$, which is simply the product action of G on $\mathrm{Ch}(I)$ and on $\mathrm{Typ}(I)$.

$$\langle D\rangle \cdot g = (D, \sigma) \cdot g = \big(p(g)(D), q(g)(\sigma)\big).$$

The action of G on $\mathrm{Typ}(I)$ defines a homomorphism of G into the set of bijections of $\mathrm{Typ}(I)$:

(3) $$1 \to G(0) \to G \xrightarrow{q} H \to 1,$$

with H contained in $S(\mathrm{Typ}(I))$.

Furthermore, we had seen in Ch. 1 that one can identify the set $\mathrm{Typ}(I)$ with $S(M(C))$ by virtue of the regular action of that latter group, which we can hence identify with the left regular action of $S(M(C))$ on itself. Moreover one sees immediately that the actions of G and of $S(M(C))$ on $\mathrm{Typ}(I)$ commute. It is well known that the commutant of right regular action of a group on itself is left regular action of the same group. Hence we can regard the exact sequence (3) as follows-

(3′) $$1 \to G(0) \to G \xrightarrow{q} \mathrm{Typ}(G) \to 1$$

and we can identify $\mathrm{Typ}(G)$ with a subgroup of $S(M(C))$.

We consider the G-equivariant projections

$$P\colon \langle \mathrm{Ch}\rangle \to \mathrm{Ch}, \quad Q\colon \langle \mathrm{Ch}\rangle \to \mathrm{Typ}(I)$$

defined by the decomposition of $\langle \mathrm{Ch}\rangle = \mathrm{Ch} \times \mathrm{Typ}(I)$ on the two factors. In fact, they are "fibred extensions" in the sense of [8], in (3′), G acts on the fibres for the projection Q, and $G(0)$ is the set of elements in G for which, for all chambers D‘ $Q(g \cdot (\langle D\rangle)) = D = Q(\langle D\rangle)$, and similarly for the pair (P, p).

Proposition 2.1. *G acts faithfully on* $\mathrm{Ch}(I)$.

We leave proof to the reader.

Definition 2.2. Let A be an apartment of I, and H a subgroup of $S(M(C))$ (thought of as a set of type mappings of the building). Then let $\langle \mathrm{Ch}(A; H)\rangle = \{\langle C\rangle | C$ is in A and $t(\langle C\rangle)$ is in $H\}$. The pair $(M(A), \langle \mathrm{Ch}(A; H)\rangle)$, will be called the *typed apartment* $\langle A; H\rangle$, H its group of types, and A its underlying abstract apartment. Set-theoretically, $\langle \mathrm{CH}(A; H)\rangle$ is the product $\mathrm{Ch}(A) \times H$.

We shall now fix a subgroup H of $\mathrm{Typ}(I)$. From now on we shall make the following assumption about our group G, which is the direct analogue of the hypotheses of Theorem 1.2 for typed apartments and non type-preserving groups.

(S) Standard hypothesis. G acts transitively on the set of pairs $(\langle C\rangle, \langle A; \mathrm{Typ}(G)\rangle)$, such that $\langle C\rangle$ is a typed chamber of type h contained in *a* typed apartment $\langle A; \mathrm{Typ}(G)\rangle$ with h in $\mathrm{Typ}(G)$.

We shall fix $\langle C\rangle$ contained in $\langle \mathrm{Ch}(A)\rangle$, and denote by B and N the stabilizers of $\langle C\rangle$ and $\langle A;H\rangle$, respectively. We shall always denote $\mathrm{Typ}(G)$ by H. Given a pair of subgroups B, N, with $B\cap N \lhd N$, $W=N/N\cap B$, we shall write right, left and double cosets $n\cdot B$, $B\cdot n$, $B\cdot n\cdot B$ unambiguously as $w\cdot B$, $B\cdot w$, etc. where w is the image of n in W under the natural projection.

The formula (B) of the following Proposition 2.2 is an obvious generalization of the Bruhat decomposition (see [3.0] for the Bruhat decomposition in its original setting, and also the references on Tits systems).

Proposition 2.2.
(i) $B\cap N\lhd N$.
(ii) *G is the disjoint union of all double cosets $B\cdot w\cdot B$, w in W:*

$$\text{(B)}\quad G=B\cdot W\cdot B.$$

Proof. We first prove that $B\cap N\lhd N$. As above, we let t be the unique type mapping defined with respect to C which is the identity on $M(C)$. Let T be the stabilizer of $\langle C\rangle$ in N. Then, by Lemma 2.1, T is contained in $\ker(q)$. We shall denote by o the homomorphism defining the action of N on $\mathrm{Ch}(A)$, i.e. the restriction of the homomorphism p to N, restricted to the invariant subset $\mathrm{Ch}(A)$. T is in the kernel of that homomorphism. On the other hand, $N\cap\ker(q)\cap\ker(o)$ is obviously contained in T. Hence T is the intersection of two normal subgroups of N, hence is normal.

We let $W=N/N\cap B$. To show (B), it suffices to show that for any $\langle D\rangle$ in $\langle\mathrm{Ch}(I)\rangle$, there exists b in B and n in N such that $\langle D\rangle=\langle C\rangle\cdot n\cdot b$. By Lemma 2.1, there exists a typed apartment $\langle A'\rangle$ containing $\langle C\rangle$ and $\langle D\rangle$. By hypothesis there exists an element b' in G which maps the pair $(\langle C\rangle,\langle A'\rangle)$ into the pair $(\langle C\rangle,\langle A\rangle)$. By definition of B, b' is in B. Applying the hypothesis again, we see that there exists an element n in G which maps the pair $(\langle C\rangle,\langle A\rangle)$ to $(\langle D\rangle\cdot b',\langle A\rangle)$. By definition of N, n is in N. Putting $b=(b')^{-1}$, $\langle D\rangle=\langle C\rangle\cdot n\cdot b$.

Using the G-equivariant projections P and Q onto $\mathrm{Ch}(A)$ and $\mathrm{Typ}(G)$ respectively, the disjointness of the decomposition (B) follows immediately from Bruhat decomposition of $G(0)$, which follows from Tits' Theorem 1.2, and the fact that $G/G(0)$ acts freely on $\mathrm{Typ}(G)$. This proves Bruhat decomposition.

For any subgroup P of G, $P(0)$ will denote the intersection of P with $G(0)$. We note that T and B are both contained in $G(0)$ by Lemma 2.1. Hence the extension

$$1\to T\to N\to W\to 1$$

restricts to a subextension

$$1\to T\to N(0)\to W(0)\to 1$$

with $W(0)$ contained in W.

Lemma 2.4. *W acts regularly on* $\langle \mathrm{CH}(A, H) \rangle$.

The proof is routine and is left to the reader.

Lemma 2.5. *Assuming* (S), *the homomorphism q restricted to N and to* $\langle \mathrm{Ch}(A, H) \rangle$ *defines an extension*

$$(4) \qquad 1 \to W(0) \to W \to H \to 1.$$

$W = W(0) \rtimes H$. There exists a splitting from H into W which maps H onto the stabilizer in W of C.

Proof. By the Standard hypothesis (S), q restricts to a surjection from N onto H. Since we have seen that T is contained in the kernel of q, q defines a surjection, still denoted by q, from W onto H, with kernel $W(0)$. We now restrict q to the stabilizer in W of C. Since, by the previous Lemma, W acts regularly on $\langle \mathrm{Ch}(A) \rangle$, this restriction is still surjective. It is clearly injective. Inverting that restriction from H onto the stabilizer, we obtain the splitting.

We note that the Standard hypothesis implies immediately that the hypotheses of Theorem 1.2 of [12] are satisfied, and hence that the quadruple of $(G(0), B(0), W(0), S)$, for some suitable S, constitute a Tits system, and hence that the following corollary is true. However, it also follows immediately from Proposition 2.2:

Proposition 2.3. $G(0) = B \cdot W(0) \cdot B$.

Proof. Writing an element in $G(0)$, as $g = b \cdot n \cdot b'$, and keeping in mind the fact that $B = B(0)$ is contained in $G(0) = \ker(q)$, we see that n also lies in $\ker(q)$, hence in $N(0)$. The result follows.

The following notion was introduced (in slightly different form) in [9].

Definition 2.3. Let G be a group, N and B two subgroups. We say that a triple (G, B, N) *gives rise to a normalizing Tits system* if and only if there exists a set S for which the axioms (T. 1), (T. 2), and (T. 3) of Definition 1.5 are verified but no such S for which (T. 4) is also verified. If S is such a set, one calls the quadruple (G, B, N, S) a normalizing Tits system.

Theorem 2.1. *Let G satisfy the hypothesis* (S), *and let B, N, W, $\langle C \rangle$, $\langle A; H \rangle$ be as above. Let $W = W(0) \rtimes H$ be the splitting of Lemma* 2.5. *Then there exists a set of generators S of $W(0)$, such that, for any set of generators R of H, $(G, B, N, R \cup S)$ form a normalizing Tits system. More precisely, all elements of R normalize B. There exists no set U of generators of W for which* (T. 4) *is also verified.*

Proof. As we have remarked, by [12], Theorem 1.2, there exists a set of generators of order 2 in $W(0)$, S for which $(G(0), B(0), N(0), S)$ forms a Tits system. We have seen that $B(0)=B$.

We choose this set S of generators of $W(0)$ and any set R of generators of H. We already know Bruhat decomposition. Hence we conclude immediately that the axioms (T. 1) and (T. 2) are satisfied. We need to verify (T. 3), i.e. we must show for each element in the generating set of W, $R\cup S$, that the condition $B\cdot w\cdot B\cdot t\cdot B= =B\cdot w\cdot t\cdot B\cup B\cdot w\cdot B$ is satisfied. In fact we shall prove that for all w in W,

(T. 3. H) $B\cdot w\cdot B\cdot h\cdot B = B\cdot w\cdot h\cdot B$ for any h in H

(T. 3. S) $B\cdot w\cdot B\cdot s\cdot B\subset B\cdot w\cdot s\cdot B\cup B\cdot w\cdot B$ for any s in S.

We consider the splitting of the extension

$$1\to W(0)\to W\to H\to 1$$

defined in Lemma 2.5: H acts regularly on the set of typed chambers (C is our fixed chamber) $\{\langle C\rangle=(C,h)|h \text{ in } H\}$. It follows from Lemma 2.1 that B is the intersection of the stabilizers in G of each of these typed chambers. Hence H normalizes B. Hence $B\cdot h=h\cdot B$, and $B\cdot W\cdot B\cdot h=B\cdot W\cdot h\cdot B$, which proves (T. 3. H).

Let $w=hw'$, then

$$B\cdot w\cdot B\cdot s\cdot B = B\cdot h\cdot w'\cdot B\cdot s\cdot B = h\cdot B\cdot w'\cdot B\cdot s\cdot B=h\cdot B\cdot w'\cdot s\cdot B\cup h\cdot B\cdot w'\cdot B=$$

$$= B\cdot w\cdot s\cdot B\cup B\cdot w\cdot B.$$

This proves (T. 3. S).

It remains to show that there exists no set U of generators of W for which (G, B, N, U) is a Tits system. This follows from the fact that B is not self-normalizing, violating a well-known property of groups with Tits system.

Example 1. Let I be projective flag variety over any field of dimension n. It is immediate to verify that the conditions of our result hold, with $\mathrm{Typ}\,(\mathrm{Aut}\,(I))= =Z/2\cdot Z$. Elements of the non-trivial coset of $G(0)$ in $\mathrm{Aut}\,(I)$ are classically known as "polarities". They lie at the basis of the duality theorem of projective geometry. (Their analogues for buildings of type $D(4)$, where $\mathrm{Typ}\,(\mathrm{Aut}\,(I))$ is the the symmetric group on 3 letters, have been extensively studied in [14].)

A splitting of the extension of $\mathrm{Aut}\,(I)\to Z/2\cdot Z$, is given by the map associating to every subspace its orthogonal complement, with respect to some inner product in the underlying space. Typed apartments are defined by unordered orthogonal bases, and consist of all ascending or descending flags formed from this basis. Given a chamber $\langle C\rangle$ in such an apartment, for instance the increasing flag defined by an ordering of that orthonormal basis in the underlying linear space, $e(1), \ldots, e(n+1)$, the splitting of the Weyl group of Lemma 2.5, would be given by the composition $t \circ s$, where t is the orthogonal complement map and s the permutation $i \to n+1-i$. It has the property that all terms of the flag are incident to their images under $t \circ s$. In particular, the "polarity" $t \circ s$ has the points corresponding to the linear subvariety generated by $\{e(i)\}, i=1, \ldots, [(n+1)/2]$, for self-adjoint points. (For an extensive discussion of self-adjoint points for groups of type $D(4)$, see [14].) The Theorem of this paper then says that the group $\mathrm{Aut}(I)=B \cdot W \cdot B$, with B the stabilizer of the flag $\langle e(0)\rangle, \ldots, (e(0), \ldots, e(i)\rangle, \ldots)$, and W the semi-direct product of the symmetric group on the basis $e(i)$ and $t \circ s$.

Example 2. I is a homogeneous tree. Then the automorphism group of I satisfies our conditions, and we find the result of [9].

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Equivalence systems and generalized wreath products

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1. Introduction

Wreath products of two groups have been constructed in various ways, and the different types of construction lead to different properties of the product (see, for example, [7]). Important types of construction are the complete and the restricted wreath product. If A is a group and B is a permutation group on a set X then the complete wreath product $A \operatorname{Wr} B$ is the semidirect product of A^X and B, where B acts on A^X by permuting the components. The restricted wreath product $A \operatorname{wr} B$ is the semidirect product of $D=\{a=(a_x)_{x\in X}\in A^X | a_x=1$ for all but finitely many $x\}$ with the action as above.

A generalization of the restricted wreath product to a set of permutation groups indexed by a totally ordered set was given by P. HALL [5], the same construction also works for a partially ordered index set (see, for example, [4], [8]). In the case of the complete wreath product there is more than one natural way to generalize it. One construction was given by W. CH. HOLLAND [6], a different one by CH. WELLS [9].

An equivalence system (X, E) is a pair consisting of a set X and a set E of equivalence relations on X. The automorphism group $\operatorname{Aut}(X, E)$ is the group of all permutations of X which leave each relation in E invariant, that is, $\operatorname{Aut}(X, E)= =\{g\in \operatorname{Sym}(X) | x\, e\, y$ if and only if $(xg)\, e\, (yg)$ for all $x, y\in X$ and $e\in E\}$. In [3], the author has shown that if (X, E) is an equivalence system with E totally ordered then $\operatorname{Aut}(X, E)$ is isomorphic to a generalized wreath product of full symmetric groups. In [2], the author considered equivalence systems where X is countable, E is totally ordered and $\operatorname{Aut}(X, E)$ is transitive on X. It should be possible to describe the automorphism groups which occur there as generalized wreath products (in a suitable sense) of full symmetric groups, however they can not be described as such products either in the sense of Holland or of Wells. This provides the motiva-

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tion to give a construction of a generalization of the complete wreath product which includes all the constructions above, and to investigate the properties of such wreath products.

2. Systematic subsets

If Λ is a partially ordered set (short: poset) and G_λ is a permutation group on a set X_λ for $\lambda\in\Lambda$ then the generalized wreath product will be a permutation group on a subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$. The constructions of Holland and Wells use different subsets of X, and in this paper we shall see that there is a still greater choice of suitable subsets. It is, however, sensible to demand that all constructions should give the same group if the index set Λ is finite, and also that an associative law like Theorem 3.8 in [6] holds. This gives a certain restriction on the kind of subset of X which we shall consider.

A subset Σ of a poset Λ is called an ideal if whenever $\sigma\in\Sigma$ and $\lambda\in\Lambda$ such that $\lambda\leqq\sigma$ then $\lambda\in\Sigma$. The dual concept is called a filter. Note that the complement of any ideal is a filter and vice versa. In the rest of this section, let Λ be a poset, let X_λ be a non-empty set for $\lambda\in\Lambda$, and let $X=\prod_{\lambda\in\Lambda} X_\lambda$. A non-empty subset S of X is called systematic if the following two conditions hold. (1) For every ideal Σ of Λ and for all $x,y\in S$ if $z\in X$ is defined by $z_\lambda=x_\lambda$ if $\lambda\in\Sigma$ and $z_\lambda=y_\lambda$ if $\lambda\notin\Sigma$ then $z\in S$. (2) For all $\lambda\in\Lambda$ and all $r\in X_\lambda$ there exists $x\in S$ such that $x_\lambda=r$. A systematic subset S of X is called strongly systematic if (1) holds for any subset Σ of Λ.

A subset Φ of Λ is called convex if whenever $\varphi_1,\varphi_2\in\Phi$ and $\lambda\in\Lambda$ such that $\varphi_1\leqq\lambda\leqq\varphi_2$ then $\lambda\in\Phi$. A non-empty subset Φ of Λ is called an order block if for all $\lambda\in\Lambda\setminus\Phi$ we have either $\lambda>\varphi$ for all $\varphi\in\Phi$ or $\lambda<\varphi$ for all $\varphi\in\Phi$ or λ and φ are incomparable for all $\varphi\in\Phi$. It is easy to see that ideals, filters and order blocks are convex. We now show that in the definition of a systematic set we could replace (1) by a stronger condition.

Lemma 2.1. *Let $S\subseteqq X$ be systematic, $\Phi\subseteqq\Lambda$ convex, and let $x,y\in S$. If $z\in X$ is defined by $z_\lambda=x_\lambda$ for $\lambda\in\Phi$ and $z_\lambda=y_\lambda$ for $\lambda\notin\Phi$ then $z\in S$.*

Proof. Let $\Sigma_1=\{\lambda\in\Lambda \mid \text{there exists } \varphi\in\Phi \text{ such that } \lambda\leqq\varphi\}$, and let $\Sigma_2=\Sigma_1\setminus\Phi$. We claim that Σ_1, Σ_2 are ideals. This is trivial for Σ_1. So let $\sigma\in\Sigma_2$, $\lambda\in\Lambda$ such that $\lambda\leqq\sigma$. Clearly $\lambda\in\Sigma_1$, and there exists $\varphi\in\Phi$ such that $\sigma\leqq\varphi$. Suppose that $\lambda\in\Phi$. Then we have $\lambda\leqq\sigma\leqq\varphi$, and as Φ is convex it follows that $\sigma\in\Phi$, giving a contradiction. Hence $\lambda\in\Sigma_1\setminus\Phi=\Sigma_2$, which proves the claim. Now let $z'\in X$ be defined by $z'_\lambda=x_\lambda$ if $\lambda\in\Sigma_1$ and $z'_\lambda=y_\lambda$ if $\lambda\notin\Sigma_1$. Then $z'\in S$. Now note that $z_\lambda=y_\lambda$ if $\lambda\in\Sigma_2$ and $z_\lambda=z'_\lambda$ if $\lambda\notin\Sigma_2$. Therefore $z\in S$.

Lemma 2.2. *Let* $S \subseteqq X$ *be systematic, and let* $x \in S$. *Then if* $x' \in X$ *is such that* $x'_\lambda = x_\lambda$ *for all but finitely many* $\lambda \in \Lambda$ *then* $x' \in S$.

This follows easily from Lemma 2.1 and condition (2) in the definition. As a consequence of Lemma 2.2 we immediately get

Lemma 2.3. *Let* $x \in X$. *Then* $S(x) := \{x' \in X | x'_\lambda = x_\lambda$ for all but finitely many $\lambda \in \Lambda\}$ *is strongly systematic, and it is the unique minimal systematic subset of* X *containing* x.

Lemma 2.4. *An intersection of systematic subsets is either empty or systematic.*

The proof of this shall be left to the reader. Note that we can define the join of any collection of systematic subsets as the intersection of all systematic subsets which contain all members of the collection (this is well defined as obviously X itself is systematic). Then the set of all systematic subsets together with the empty set forms a complete lattice under set-theoretic intersection and the join as defined.

If Φ is any subset of the poset Λ and X_λ are non-empty sets for $\lambda \in \Lambda$ then there exists a canonic projection p_Φ from $X = \prod_{\lambda \in \Lambda} X_\lambda$ onto $\prod_{\lambda \in \Phi} X_\lambda$ where $p_\Phi(x) = (x_\lambda)_{\lambda \in \Phi}$. Note that if P is a partition of Λ then we get a natural bijection $X \to \prod_{\Phi \in P} \prod_{\lambda \in \Phi} X_\lambda$ by $x \mapsto (p_\Phi(x))_{\Phi \in P}$. With this notation we can see that a subset S of X is systematic if and only if the set S is mapped onto $p_\Phi(S) \times p_{\Lambda \setminus \Phi}(S)$ for every convex subset $\Phi \subseteqq \Lambda$ under the natural bijection $X \to p_\Phi(X) \times p_{\Lambda \setminus \Phi}(X)$ and $p_{\{\lambda\}}(S) = X_\lambda$ for all $\lambda \in \Lambda$. Also S is strongly systematic if and only if the above holds for every subset $\Phi \subseteqq \Lambda$ and $p_{\{\lambda\}}(S) = X_\lambda$ for all $\lambda \in \Lambda$.

3. Definition and elementary properties of the wreath product

Let Λ be a poset, let G_λ be a permutation group on a non-empty set X_λ for $\lambda \in \Lambda$, and let S be a systematic subset of $X = \prod_{\lambda \in \Lambda} X_\lambda$. For all $\lambda \in \Lambda$ we define equivalence relations $e(\lambda)$ and $e_L(\lambda)$ on S in the following way. Whenever $x, y \in S$ we have $x\, e(\lambda)\, y$ if and only if $x_\mu = y_\mu$ for all $\mu > \lambda$ and $x\, e_L(\lambda)\, y$ if and only if $x_\mu = y_\mu$ for all $\mu \geqq \lambda$. We let $E = \{e(\lambda), e_L(\lambda) | \lambda \in \Lambda\}$. Then we define the wreath product $S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ by $S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda = \{g \in \mathrm{Aut}\,(S, E) |$For all $x \in S$ and $\lambda \in \Lambda$ there exists $g_{\lambda, x} \in G_\lambda$ such that $(x'g)_\lambda = x' g_{\lambda, x}$ for all $x' \in S$ with $x'\, e(\lambda)\, x\}$.

Note that it is easy to see that $S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ is, in fact, a group, and that $(g^{-1})_{\lambda, x} = (g_{\lambda, xg^{-1}})^{-1}$, and $(gh)_{\lambda, x} = g_{\lambda, x} h_{\lambda, xg}$. Also note that $\mathrm{Aut}\,(S, E) = S - \mathrm{WR}_{\lambda \in \Lambda} \mathrm{Sym}\,(X_\lambda)$.

We say that a subset Φ of a poset Λ satisfies the maximal condition if every non-empty subset of Φ has a maximal element. If X_λ is a non-empty set for $\lambda \in \Lambda$

and $x \in X = \prod_{\lambda \in \Lambda} X_\lambda$ then let $H(x) = \{y \in X \mid \{\lambda \in \Lambda \mid y_\lambda \neq x_\lambda\}$ satisfies the maximal condition$\}$. It is not hard to see that $H(x)$ is strongly systematic. Then Holland's wreath products [6] are just the products $H(x)-\mathrm{WR}_{\lambda \in \Lambda} G_\lambda$. Wells's wreath products [9] are the products $X-\mathrm{WR}_{\lambda \in \Lambda} G_\lambda$, and the groups studied by the author in [2] are of the form $S(x)-\mathrm{WR}_{\lambda \in \Lambda} \mathrm{Sym}(X_\lambda)$. Therefore the construction we have given generalizes all those wreath products. Note that if Λ is finite then by Lemma 2.3 we get $S(x) = X$ for all $x \in X$, thus there is only one systematic subset, and hence only one wreath product. In particular, if $\Lambda = \{1, 2\}$ with the natural order, we get the ordinary complete wreath product.

4. The associative law

We shall now investigate in which way wreath products over large index sets Λ can be put together from products over certain subsets of Λ. In order to do this we need more properties of systematic subsets.

Lemma 4.1. *Let S be a systematic subset of X, and let Φ be a subset of Λ. Then $p_\Phi(S)$ is a systematic subset of $p_\Phi(X) = \prod_{\lambda \in \Phi} X_\lambda$.*

Proof. Let Σ be an ideal of Φ, and let $x, y \in p_\Phi(S)$, let $z \in p_\Phi(X)$ be defined by $z_\lambda = x_\lambda$ if $\lambda \in \Sigma$ and $z_\lambda = y_\lambda$ if $\lambda \notin \Sigma$. Now let $\bar{x}, \bar{y} \in S$ such that $x = p_\Phi(\bar{x})$, $y = p_\Phi(\bar{y})$, and let $\bar{\Sigma} = \{\lambda \in \Lambda \mid$there exists $\sigma \in \Sigma$ such that $\lambda \leqq \sigma\}$. Clearly $\bar{\Sigma}$ is an ideal of Λ. Now let $\bar{z} \in X$ be defined by $\bar{z}_\lambda = \bar{x}_\lambda$ if $\lambda \in \bar{\Sigma}$ and $\bar{z}_\lambda = \bar{y}_\lambda$ if $\lambda \notin \bar{\Sigma}$. Then $\bar{z} \in S$. It remains to prove that $p_\Phi(\bar{z}) = z$. Let $\lambda \in \Phi$. First suppose that $\lambda \notin \bar{\Sigma}$. Then also $\lambda \notin \Sigma$, and we have $\bar{z}_\lambda = \bar{y}_\lambda = y_\lambda = z_\lambda$. Now suppose that $\lambda \in \bar{\Sigma}$. Then there exists $\sigma \in \Sigma$ such that $\lambda \leqq \sigma$. But $\lambda \in \Phi$ and Σ is an ideal of Φ, hence $\lambda \in \Sigma$. But then $\bar{z}_\lambda = \bar{x}_\lambda = x_\lambda = z_\lambda$. Therefore we have $z = p_\Phi(\bar{z})$, and $p_\Phi(S)$ is systematic, as condition (2) is trivially fulfilled.

Lemma 4.2. *Let P be a partition of a poset Λ into order blocks. Let $\Phi_1, \Phi_2 \in P$. Then the following are equivalent:*

(i) *There exist $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ such that $\varphi_1 \leqq \varphi_2$.*

(ii) *Either $\Phi_1 = \Phi_2$ or for all $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ we have $\varphi_1 < \varphi_2$.*

The proof of this is easy. As a consequence, we can define a partial order on P in the following way. If $\Phi_1, \Phi_2 \in P$ then $\Phi_1 \leqq \Phi_2$ if and only if there exist $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ such that $\varphi_1 \leqq \varphi_2$.

Lemma 4.3. *Let P be a partition of Λ into order blocks, and let S be a systematic subset of X. Then $\bar{S} := \{(p_\Phi(x))_{\Phi \in P} \mid x \in S\}$ is a systematic subset of $\bar{X} := \prod_{\Phi \in P} p_\Phi(S)$.*

Proof. Let I be an ideal of P, let $x, y \in \bar{S}$, and let $z \in \bar{X}$ be defined by $z_\Phi = x_\Phi$ for $\Phi \in I$ and $z_\Phi = y_\Phi$ for $\Phi \notin I$. Now let $x', y' \in S$ such that $x_\Phi = p_\Phi(x')$ and $y_\Phi = p_\Phi(y')$ for all $\Phi \in P$. Note that $\Sigma := \{\lambda \mid \text{there exists } \Phi \in I \text{ such that } \lambda \in \Phi\}$ is an ideal of Λ, and if $z' \in X$ is defined by $z'_\lambda = x'_\lambda$ if $\lambda \in \Sigma$ and $z'_\lambda = y'_\lambda$ if $\lambda \notin \Sigma$ then $z' \in S$. But then clearly $z_\Phi = p_\Phi(z')$ for all $\Phi \in P$, and hence $z \in \bar{S}$, thus we have condition (1). Condition (2) is satisfied by definition.

Theorem 4.4. *Let P be a partition of a poset Λ into order blocks. For $\lambda \in \Lambda$ let G_λ be a permutation group on a non-empty set X_λ, and let S be a systematic subset of $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $\bar{S} = \{(p_\Phi(x))_{\Phi \in P} \mid x \in S\}$. Then $S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ is permutationally isomorphic to $\bar{S} - \mathrm{WR}_{\Phi \in P}\,(p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda)$.*

Proof. We have a partial order on P by Lemma 4.2. The set $p_\Phi(S)$ is a systematic subset of $\prod_{\lambda \in \Phi} X_\lambda$ by Lemma 4.1, and $\bar{S}$ is a systematic subset of $\bar{X} = \prod_{\Phi \in P} p_\Phi(S)$ by Lemma 4.3. Therefore the group $\bar{G} = \bar{S} - \mathrm{WR}_{\Phi \in P}\,(p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda)$ is well-defined. Let $\beta \colon X \to \prod_{\Phi \in P} \prod_{\lambda \in \Phi} X_\lambda$ be the natural bijection, note that $\beta(S) = \bar{S}$, and denote $G = S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$. We define a mapping $\varphi \colon G \to \mathrm{Sym}(\bar{S})$ by $y(g\varphi) = y\beta^{-1} g \beta$. Clearly φ is a monomorphism and G is permutationally isomorphic to its image under φ. Thus it remains to prove that $\varphi(G) = \bar{G}$.

It is easy to see that $g\varphi$ leaves the relations $e(\Phi)$ and $e_L(\Phi)$ invariant for $\Phi \in P$ and $g \in G$. Note that if $x \in S$, $g \in G$ then $(x\beta)(g\varphi) = xg\beta$.

Let $\beta(x) \in \bar{S}$, and let $\Phi \in P$. We claim that there exists $g_{\Phi,x} \in p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$ such that $p_\Phi(x'g) = p_\Phi(x') g_{\Phi,x}$ for all $x' \in S$ with $p_\Phi(x') e(\Phi) p_\Phi(x)$. We define $g_{\Phi,x}$ in the following way. Let $y \in p_\Phi(S)$, and let $\bar{y} \in S$ be such that $y = p_\Phi(\bar{y})$. Then from Lemma 2.1 it follows that if $z_x(y) \in X$ is defined by $z_x(y)_\lambda = \bar{y}_\lambda$ for $\lambda \in \Phi$ and $z_x(y)_\lambda = x_\lambda$ for $\lambda \notin \Phi$ then $z_x(y) \in S$ and $p_\Phi(z_x(y)) = y$. Also note that $z_x(y)$ is uniquely defined with respect to this property and independent of the choice of $\bar{y}$. Then let $y g_{\Phi,x} = p_\Phi(z_x(y) g)$. Now let $\lambda \in \Phi$ and $y' \in p_\Phi(S)$ such that $y' \, e(\lambda) \, y$ (where the relation $e(\lambda)$ is taken in $p_\Phi(S)$). Then let $\bar{g}_{\lambda,y} := g_{\lambda, z_x(y)}$. We then note that we get $(y' g_{\Phi,x}) = y' \bar{g}_{\lambda,y}$. Hence we have $g_{\Phi,x} \in p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$, and we have $p_\Phi(x'g) = p_\Phi(x') g_{\Phi,x}$, which establishes the claim.

Therefore φ maps G into $\bar{G}$, and it remains to prove that φ is surjective. Let $\bar{g} \in \bar{G}$. If $x \in X$ then we define g by $xg := x\beta\bar{g}\beta^{-1}$. Then clearly $g \in \mathrm{Sym}(X)$. Now let $\lambda \in \Lambda$ and $x, x' \in S$. Let $\Phi \in P$ be such that $\lambda \in \Phi$. Then $\{\mu \in \Lambda \mid \mu > \lambda\} = \{\mu \in \Lambda \mid \text{there exists } \Phi' \in P,\ \Phi' > \Phi \text{ and } \mu \in \Phi'\} \cup \{\mu \in \Phi \mid \mu > \lambda\}$. By definition of $\bar{G}$, it follows that $x_\mu = x'_\mu$ is equivalent to $(xg)_\mu = (x'g)_\mu$ for all $\mu \in \Lambda$ such that there exists $\Phi' \in P$, $\Phi' > \Phi$ with $\mu \in \Phi'$. But then by definition of $p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$ we get that $x_\mu = x'_\mu$ is equivalent to $(xg)_\mu = (x'g)_\mu$ for all $\mu \in \Phi$ with $\mu > \lambda$. Therefore g leaves $e(\lambda)$ invariant. Similarly, it also leaves $e_L(\lambda)$ invariant. Now let $x \in S$, $\lambda \in \Lambda$. We finally have to show that there exists $g_{\lambda,x} \in G_\lambda$ such that

$(x'g)_\lambda = x'g_{\lambda,x}$ for all $x' \in S$ with $x' e(\lambda) x$. We know that if $\Phi \in P$ such that $\lambda \in \Phi$ then there exists $g_{\Phi,x} \in p_\Phi(S) - \mathrm{WR}_{\lambda\in\Phi} G_\lambda$ such that $(x'g\beta)_\Phi = p_\Phi(x')g_{\Phi,x}$ for all $x'\beta\, e(\Phi)\, x\beta$ (note that $x' e(\lambda) x$ implies $x'\beta\, e(\Phi)\, x\beta$). But then we know that there exists $g_{\lambda,x} \in G_\lambda$ such that $(y' g_{\Phi,x})_\lambda = y'_\lambda g_{\lambda,x}$ for all $y' \in p_\Phi(S)$ with $y' e(\lambda) p_\Phi(x)$ (where $e(\lambda)$ is the relation in $p_\Phi(S)$). But note that if $x' e(\lambda) x$ (with $e(\lambda)$ in S) then also $p_\Phi(x')\, e(\lambda)\, p_\Phi(x)$ (with $e(\lambda)$ in $p_\Phi(S)$). Together we get that if $x' e(\lambda) x$ then $(x'g)_\lambda = p_\Phi(x')_\lambda g_{\lambda,x} = x'_\lambda g_{\lambda,x}$, which proves the theorem.

5. Embeddings and transitivity

We now want to see which wreath products are transitive. We first consider products of full symmetric groups.

Theorem 5.1. *Let Λ be a partially ordered set, let X_λ be a non-empty set for $\lambda \in \Lambda$, and let S be a strongly systematic subset of $X = \prod_{\lambda\in\Lambda} X_\lambda$. Then* $\mathrm{Aut}(S, E)$ *is transitive on S.*

Proof. Let $x, y \in S$. Define $g\colon S \to X$ in the following way. If $z \in S$ then

$$(zg)_\lambda = \begin{cases} y_\lambda & \text{if } \ z_\lambda = x_\lambda \\ x_\lambda & \text{if } \ z_\lambda = y_\lambda \\ z_\lambda & \text{otherwise.} \end{cases}$$

First note that g maps S into S, as S is strongly systematic. Next, clearly g^2 is the identity on S, and hence $g \in \mathrm{Sym}(S)$. Finally, it is obvious that g leaves the equivalence relations $e(\lambda)$ and $e_L(\lambda)$ invariant, hence $g \in \mathrm{Aut}(S, E)$. As $xg = y$, we get the transitivity of $\mathrm{Aut}(S, E)$.

Clearly for the transitivity of the wreath product of groups G_λ it is necessary that all G_λ are transitive. This, however, is not a sufficient condition.

Proposition 5.2. *There exist a poset Λ and transitive permutation groups G_λ on sets X_λ ($\lambda \in \Lambda$), and a strongly systematic subset S of $X = \prod_{\lambda\in\Lambda} X_\lambda$ such that $G := S - \mathrm{WR}_{\lambda\in\Lambda} G_\lambda$ is not transitive on S.*

Proof. Let $\Lambda = \mathbf{Z}$ with the trivial order (i.e. any two distinct elements are incomparable), let $X_\lambda = \{0, 1, 2\}$ and let G_λ be generated by the cyclic permutation (012) for $\lambda \in \Lambda$. Let $S = \{x \in \prod_{\lambda\in\Lambda} X_\lambda \mid x_\lambda \neq 2$ for all but finitely many $\lambda\}$. Let $u, v \in S$ be defined by $u_\lambda = 0$ and $v_\lambda = 1$ for all $\lambda \in \Lambda$. Suppose that G is transitive on S. Then there exists $g \in G$ such that $ug = v$. Now note that $(xg)_\lambda = x_\lambda g_{\lambda,u}$ for all $x \in S$ with $x\, e(\lambda)\, u$. In particular, we have $1 = v_\lambda = (ug)_\lambda = u_\lambda g_{\lambda,u} = 0 g_{\lambda,u}$. But then we must have $g_{\lambda,u} = (012)$ for all $\lambda \in \Lambda$. Now we also have $v\, e(\lambda)\, u$ for all $\lambda \in \Lambda$.

Hence it follows that $(vg)_\lambda = v_\lambda g_{\lambda,u} = 1(012) = 2$ for all $\lambda \in \Lambda$. But then $vg \notin S$, which is a contradiction. Therefore G is not transitive.

In contrast to this result, for some systematic subsets S the wreath product is always transitive whenever all groups G_λ are transitive. This is trivial for $S=X$, it holds for $S=H(x)$ (Thm. 3.9 in [6]), and also for $S=S(x)$ (as then the restricted wreath product is a subgroup which is already transitive). If S and T are systematic subsets with $S \subseteqq T$ then it is natural to ask if the wreath product constructed on S can be embedded in a natural way into the wreath product constructed on T. However, this does not need to be the case in general.

Proposition 5.3. *There exists a poset Λ, non-empty sets X_λ for $\lambda \in \Lambda$ and systematic subsets $S, T \subseteqq X = \prod_{\lambda \in \Lambda} X_\lambda$ with $S \subseteqq T$ such that there does not exist a monomorphism φ:* $\mathrm{Aut}(S,E) \to \mathrm{Aut}(T,E)$ *such that $x(g\varphi) = xg$ for all $x \in S$, $g \in$* $\mathrm{Aut}(S,E)$.

Proof. Let $\Lambda = \mathbf{Z}$ with its natural order, let $X_z = \{0, 1, 2\}$ for $z \in \mathbf{Z}$. Let $X = \prod_{z \in \mathbf{Z}} X_z$, and let $S = \{x \in X \mid \text{there exist } z_U, z_L \in \mathbf{Z} \text{ such that } x_z = x_{z_U} \text{ for } z \geqq z_U$ and $x_z = x_{z_L}$ for $z \leqq z_L\}$, and $T = \{x \in X \mid \text{there exist } z_U, z_L \in \mathbf{Z} \text{ such that } x_z = x_{z_U}$ for $z \geqq z_U$ and $x_z = 2$ for $z \leqq z_L$ or $x_z \in \{0, 1\}$ for $z \leqq z_L\}$. Clearly, the sets S and T are both systematic. Let $G = \mathrm{Aut}(S,E)$ and $H = \mathrm{Aut}(T,E)$. Let $g \in G$ be defined by $(xg)_z = x_z(012)$ for all $x \in S$, $z \in \mathbf{Z}$. First note that, in fact, we have $g \in G$.

Suppose there exists a monomorphism $\varphi\colon G \to H$ such that $x(g\varphi) = xg$ for all $x \in S$, $g \in G$. Now let $x \in X$ be defined in the following way. Let $x_z = 1$ if $z < 0$ and $z \equiv 1 \pmod 2$ and let $x_z = 0$ otherwise. Note that $x \in T$. For $m \in \mathbf{Z}$ define $y(m) \in X$ in the following way. Let $y(m)_z = x_z$ if $z \geqq m$ and $y(m)_z = 0$ if $z < m$. Then clearly $y(m) \in S$ for all $m \in \mathbf{Z}$. Now $y(m)\, e(m-1)\, x$. Next note that with g defined as above we have $(y(m)g)_m = y(m)_m(012)$. Hence we must have $(x(g\varphi))_m = (y(m)(g\varphi))_m = (y(m)g)_m = y(m)_m(012)$ for all $m \in \mathbf{Z}$. But then $(x(g\varphi))_m = 2$ if $m < 0$ and $m \equiv 1 \pmod 2$ and $(x(g\varphi))_m = 1$ otherwise. But then $x(g\varphi) \notin T$, which is a contradiction.

6. Some wreath products of full symmetric groups

In this section we shall show how some wreath products of full symmetric groups can be decomposed into simpler products.

Lemma 6.1. *Let Λ be a poset, let X_λ be a non-empty set for $\lambda \in \Lambda$, and let $x \in X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $M(x) = \{y \in X \mid$ for all $\lambda \in \Lambda$ there exists $\mu \in \Lambda$ with $\mu \geqq \lambda$ such that $y_\gamma = x_\gamma$ for all $\gamma > \mu\}$. Then $M(x)$ is a systematic subset of X.*

The proof of this is similar to some proofs already given and shall therefore be omitted. Clearly, if $y \in M(x)$ then $M(y)=M(x)$, hence these sets $M(x)$ form a partition of X. We recall that a poset Λ is called upper directed if for $\lambda, \lambda' \in \Lambda$ there exists $\mu \in \Lambda$ such that $\mu \geqq \lambda$ and $\mu \geqq \lambda'$.

Theorem 6.2. *Let Λ be an upper directed poset and X_λ a non-empty set for $\lambda \in \Lambda$. Let S be a strongly systematic subset of $X = \prod_{\lambda \in \Lambda} X_\lambda$, let $C = \{M(y) | y \in S\}$ and $x \in S$. Then* $\mathrm{Aut}(S, E)$ *is permutationally isomorphic to* $\mathrm{Aut}(S \cap M(x), E) \mathrm{Wr} \,\mathrm{Sym}(C)$.

Proof. For $y \in S$ we define $M_S(y) = M(y) \cap S$. Choose a subset R of S with $x \in R$ and such that $C = \{M(r) | r \in R\}$ and $|R \cap M(y)| = 1$ for all $y \in S$. For each $r \in R$ we define a mapping $\alpha_{M(r)}$: $M_S(x) \to M_S(r)$ in the following way. Let $z \in M_S(x)$. Then

$$(z\alpha_{M(r)})_\lambda = \begin{cases} r_\lambda & \text{if} \quad z_\lambda = x_\lambda \\ x_\lambda & \text{if} \quad z_\lambda = r_\lambda \\ z_\lambda & \text{otherwise.} \end{cases}$$

We first have to show that $\alpha_{M(r)}$ maps $M_S(x)$ indeed into $M_S(r)$. It is clear that $z\alpha_{M(r)} \in S$ as S is strongly systematic. So let $\lambda \in \Lambda$. As $z \in M(x)$ there exists $\mu \in \Lambda$ with $\mu \geqq \lambda$ such that $z_\gamma = z_\gamma$ for all $\gamma > \mu$. But then we have $(z\alpha_{M(r)})_\gamma = r_\gamma$ for all $\gamma > \mu$, and hence $z\alpha_{M(r)} \in M(r)$, and also $z\alpha_{M(r)} \in M_S(r)$. We now claim that $\alpha_{M(r)}$ is a bijection. It is easy to see that it is injective. Let $s \in M_S(r)$. Define $\bar{s} \in X$ by

$$\bar{s}_\lambda = \begin{cases} r_\lambda & \text{if} \quad s_\lambda = x_\lambda \\ x_\lambda & \text{if} \quad s_\lambda = r_\lambda \\ s_\lambda & \text{otherwise.} \end{cases}$$

Then, as above, it follows that $\bar{s} \in M_S(x)$, and it is clear that $\bar{s}\alpha_{M(r)} = s$, which establishes the claim.

Let $\bar{S} = M_S(x) \times C$. We define a mapping α: $\bar{S} \to S$ by $(z, M(y))\alpha = z\alpha_{M(y)}$ for $z \in M_S(x), M(y) \in C$. We note that α is bijective. Let $X_\tau := C$ and let $\bar{\Lambda} = \Lambda \cup \{\tau\}$ where $\tau > \lambda$ for all $\lambda \in \Lambda$. Then $\bar{S}$ is a strongly systematic subset of $\bar{X} := \prod_{\lambda \in \bar{\Lambda}} X_\lambda$. Let $\bar{E}$ be the set of relations induced by E on $M_S(x)$ together with $e(\tau)$ and $e_L(\tau)$. Then by Theorem 4.4 it follows that $\mathrm{Aut}(\bar{S}, \bar{E})$ is permutationally isomorphic to $\mathrm{Aut}(M_S(x)) \mathrm{Wr}\, \mathrm{Sym}(C)$. So it remains to prove that $\mathrm{Aut}(\bar{S}, \bar{E})$ and $\mathrm{Aut}(S, E)$ are permutationally isomorphic.

We define a mapping a: $\mathrm{Aut}(S, E) \to \mathrm{Sym}(\bar{S})$ in the following way. If $\sigma \in \mathrm{Aut}(S, E)$, $(z, M(y)) \in \bar{S}$ then $(z, M(y))(\sigma a) = (z, M(y))\alpha\sigma\alpha^{-1}$. Note that it is clear that a is a monomorphism and that $\mathrm{Aut}(S, E)$ is permutationally isomorphic to its image under a. Thus all we have to prove now is that $\mathrm{Aut}(\bar{S}, \bar{E})$ is equal to the image of $\mathrm{Aut}(S, E)$ under a.

First we want to show that $\sigma a \in \mathrm{Aut}(\bar{S}, \bar{E})$ for all $\sigma \in \mathrm{Aut}(S, E)$. Let $\lambda \in \Lambda$ and let $(z, M(y))\, e(\lambda)\, (z', M(y'))$. It follows that $M(y)=M(y')$ and also $z\, e(\lambda)\, z'$. Therefore we get $(z\alpha_{M(y)})\, e(\lambda)\, (z'\alpha_{M(y)})$ and $(z\alpha_{M(y)}\sigma)\, e(\lambda)\, (z'\alpha_{M(y)}\sigma)$. As Λ is upper directed, it follows that $M(z\alpha_{M(y)}\sigma)=M(z'a_{M(y)}\sigma)$, and hence we also get $(z\alpha_{M(y)}\sigma\alpha^{-1}_{M(z\alpha_{M(y)}\sigma)})\, e(\lambda)\, (z'\alpha_{M(y)}\sigma\alpha^{-1}_{M(z'\alpha_{M(y)}\sigma)})$, and hence

$$((z, M(y))(\sigma a))\, e(\lambda)\, ((z', M(y'))(\sigma a)).$$

The converse, and the result for $e_L(\lambda)$ follow similarly. Note that $e(\tau)$ is the universal relation. Let $(z, M(y))\, e_L(\tau)\, (z', M(y'))$. This means that $M(y)=M(y')$, and as above we get $M(z\alpha_{M(y)}\sigma)=M(z'\alpha_{M(y)}\sigma)$, and hence

$$((z, M(y))(\sigma a))\, e_L(\tau)\, ((z', M(y'))(\sigma a)).$$

Again, the converse follows similarly, and hence $\sigma a \in \mathrm{Aut}(\bar{S}, \bar{E})$.

Finally we have to show that a is surjective. Let $\varrho \in \mathrm{Aut}(\bar{S}, \bar{E})$. Then we have to prove that $\alpha^{-1}\varrho\alpha \in \mathrm{Aut}(S, E)$. Let $z, z' \in S$, $\lambda \in \Lambda$ with $z\, e(\lambda)\, z'$. As Λ is upper directed, we have $M(z)=M(z')$, and as $z\alpha^{-1}=(z\alpha^{-1}_{M(z)}, M(z))$, we have $z\alpha^{-1}\, e(\lambda)\, z'\alpha^{-1}$, and therefore $z\alpha^{-1}\varrho\, e(\lambda)\, z'\alpha^{-1}\varrho$. Then again it follows that $z\alpha^{-1}\varrho\alpha\, e(\lambda)\, z'\alpha^{-1}\varrho\alpha$. The converse follows similarly, and so does the result for $e_L(\lambda)$. Therefore a is surjective, which concludes the proof of the theorem.

7. The normal structure of wreath products

We recall that the set of all ideals of a poset Λ is a complete distributive lattice with respect to set-theoretic intersection and union. If S is a systematic subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$ and Σ is an ideal of Λ then we can define an equivalence relation $e(\Sigma)$ on S by $x\, e(\Sigma)\, x'$ if and only if $x_\lambda = x'_\lambda$ for all $\lambda \notin \Sigma$. Note that $e(\Sigma)$ is the infimum over all the relations $e_L(\lambda)$ with $\lambda \notin \Sigma$.

Proposition 7.1. *Let Λ be a poset and G_λ a permutation group on a non-empty set X_λ for $\lambda \in \Lambda$. Let S be a systematic subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$, and let $G=S-\mathrm{WR}_{\lambda\in\Lambda} G_\lambda$. For every ideal Σ of Λ let $D(\Sigma)=\{g\in G \mid x\, e(\Sigma)\, xg$ for all $x \in S\}$. Then $D(\Sigma)$ is a normal subgroup of G and the mapping $\Sigma \mapsto D(\Sigma)$ is a monomorphism from the lattice of ideals of Λ into the normal subgroup lattice of G preserving arbitrary meets and finite joins.*

Proof. Trivially, $D(\Sigma)$ is a subgroup of G. Let $h \in G$, $g \in D(\Sigma)$. Note that h leaves all relations $e_L(\lambda)$ invariant, and hence also the relation $e(\Sigma)$. So if $x \in S$ then $(xh^{-1})\, e(\Sigma)\, (xh^{-1})g$, and hence also $(xh^{-1})h\, e(\Sigma)\, ((xh^{-1})g)h$, therefore

$x\,e(\Sigma)\,x(h^{-1}gh)$, and $h^{-1}gh\in D(\Sigma)$. Note that it is also trivial that if $\Sigma\subseteq\Sigma'$ then $D(\Sigma)\leqq D(\Sigma')$.

Let Σ_i $(i\in I)$ be a set of ideals. Then we have $D(\bigcap_{i\in I}\Sigma_i)\leqq D(\Sigma_j)$ for all $j\in I$, and hence $D(\bigcap_{i\in I}\Sigma_i)\leqq\bigcap_{i\in I}D(\Sigma_i)$. Conversely, let $g\in\bigcap_{i\in I}D(\Sigma_i)$. Then for all $i\in I$, $x\in S$ and all $\lambda\notin\Sigma_i$ we have $x_\lambda=(xg)_\lambda$. Hence, for all $x\in S$ and all $\lambda\notin\Sigma_i$ we have $x_\lambda=(xg)_\lambda$, therefore $x\,e(\bigcap_{i\in I}\Sigma_i)\,xg$ for all $x\in S$, and $g\in D(\bigcap_{i\in I}\Sigma_i)$.

Let Σ_1, Σ_2 be ideals and $g\in D(\Sigma_1\cup\Sigma_2)$. We define $h, h'\colon S\to X$ in the following way. If $x\in S$, $\lambda\in\Lambda$ then $(xh)_\lambda=x_\lambda g_{\lambda,x}$ if $\lambda\in\Sigma_1\setminus\Sigma_2$ and $(xh)_\lambda=x_\lambda$ otherwise. Also $(xh')_\lambda=x_\lambda g_{\lambda,xh^{-1}}$ if $\lambda\in\Sigma_2$ and $(xh')_\lambda=x_\lambda$ otherwise. First we claim that $h\in\mathrm{Sym}\,(S)$. Note that $(xh)_\lambda=(xg)_\lambda$ if $\lambda\in\Sigma_1\setminus\Sigma_2$ and $(xh)_\lambda=x_\lambda$ otherwise, hence $xh\in S$, as S is systematic. Also h is clearly injective. Let $y\in S$, and let $y'\in X$ be defined by $y'_\lambda=(yg^{-1})_\lambda$ if $\lambda\in\Sigma_1\setminus\Sigma_2$ and $y'_\lambda=y_\lambda$ otherwise. Then $y'\in S$, and $y'h=y$. Thus $h\in\mathrm{Sym}\,(S)$. Furthermore, it is not hard to see that $h\in D(\Sigma_1)$. Note that $h_{\lambda,x}=1$ if $\lambda\notin\Sigma_1\setminus\Sigma_2$ and $h_{\lambda,x}=g_{\lambda,x}$ if $\lambda\in\Sigma_1\setminus\Sigma_2$.

Next we show that $h'\in\mathrm{Sym}\,(S)$. For this, we observe that $(xh')_\lambda=(xh^{-1}g)_\lambda$ if $\lambda\in\Sigma_2$ and $(xh')_\lambda=x_\lambda$ otherwise, hence we have $xh'\in S$. As above, it follows that $h'\in D(\Sigma_2)$, and note that $h'_{\lambda,x}=1$ if $\lambda\notin\Sigma_2$ and $h'_{\lambda,x}=g'_{\lambda,xh^{-1}}$ if $\lambda\in\Sigma_2$.

Finally, we show that $g=hh'$. Let $x\in S$, $\lambda\in\Lambda$. If $\lambda\notin\Sigma_1\cup\Sigma_2$ then $(xhh')_\lambda=$ $=(xh)_\lambda=x_\lambda=(xg)_\lambda$. If $\lambda\in\Sigma_1\setminus\Sigma_2$ then $(xhh')_\lambda=(xh)_\lambda=x_\lambda g_{\lambda,x}=(xg)_\lambda$, and if $\lambda\in\Sigma_2$ then $(xhh')_\lambda=(xh_\lambda g_{\lambda,(xh)h^{-1}}=x_\lambda g_{\lambda,x}=(xg)_\lambda$. Therefore $g=hh'$, which proves the proposition.

We shall finally show that the normal subgroups constructed in Proposition 7.1 are themselves generalized wreath products. We remark that a similar result holds for generalized restricted wreath products (Thm. 4.2 in [1]). Let Λ be a poset, G_λ a permutation group on the non-empty set X_λ for $\lambda\in\Lambda$, let S be a systematic subset of $X=\prod_{\lambda\in\Lambda}X_\lambda$, and let Σ be an ideal of Λ and $D(\Sigma)$ defined as in Proposition 7.1. For $\sigma\in\Sigma$ define $F(\sigma)=\{\lambda\in\Lambda|\lambda>\sigma$ and $\lambda\notin\Sigma\}$. Let $\bar\Sigma=\{(\sigma,y)|\sigma\in\Sigma,\ y\in p_{F(\sigma)}(S)\}$. We partially order $\bar\Sigma$ by $(\sigma_1,y_1)<(\sigma_2,y_2)$ if and only if $\sigma_1<\sigma_2$ and $(y_1)_\lambda=(y_2)_\lambda$ for all $\lambda\in F(\sigma_2)$. Let $G_{(\sigma,y)}=G_\sigma$ and $X_{(\sigma,y)}=X_\sigma$ for all $y\in p_{F(\sigma)}(S)$ and $\sigma\in\Sigma$. Note that if $w\in p_{\Lambda\setminus\Sigma}(S)$ and $T(w)=\{(\sigma,y)\in\bar\Sigma|y_\lambda=w_\lambda$ for all $\lambda\in F(\sigma)\}$ then the mapping $(\sigma,y)\mapsto\sigma$ is an order-isomorphism $T(w)\to\Sigma$. Also note that this order-isomorphism induces a canonical bijection $\beta_w\colon\prod_{(\sigma,w)\in T(w)}X_{(\sigma,w)}\to\prod_{\sigma\in\Sigma}X_\sigma$. Let $\bar X=\prod_{(\sigma,y)\in\bar\Sigma}X_{(\sigma,y)}$ and $\bar S=\{x\in\bar X|$ for all $w\in p_{\Lambda\setminus\Sigma}(S)$ it follows that $\beta_w(p_{T(w)}(x))\in p_\Sigma(S)\}$.

Theorem 7.2. *Given the notations and assumptions of the preceding paragraph, the set $\bar S$ is a systematic subset of $\bar X$, and $D(\Sigma)$ is isomorphic to $\bar S-\mathrm{WR}_{(\sigma,y)\in\bar\Sigma}\,G_{(\sigma,y)}$.*

Proof. We first show that $\bar S$ is systematic. Note that if $x\in S$ we can define $\bar x\in\bar X$ by $\bar x_{(\sigma,y)}=x_\sigma$ for all $\sigma\in\Sigma$, $y\in p_{F(\sigma)}(S)$. Then clearly $\bar x\in\bar S$. So we easily get condition (2). We now show (1). Let Δ be an ideal of $\bar{\Sigma}$, let $x, y \in \bar{S}$ and define $z \in \bar{X}$ by $z_{(\sigma, v)} = x_{(\sigma, v)}$ if $(\sigma, v) \in \Delta$ and $z_{(\sigma, v)} = y_{(\sigma, v)}$ otherwise. Now let $w \in p_{\Lambda \setminus \Sigma}(S)$, and consider $\beta_w(p_{T(w)}(z))$. We have $\beta_w(p_{T(w)}(x)),\ \beta_w(p_{T(w)}(y)) \in p_\Sigma(S)$. Note that if $\bar{\Delta} = \{\sigma \in \Sigma | (\sigma, v) \in \Delta$ where $v_\lambda = w_\lambda$ for all $\lambda \in F(\sigma)\}$ then clearly $\bar{\Delta}$ is an ideal of Σ, and hence also of Λ. Also note that we have $(\beta_w(p_{T(w)}(z)))_\sigma = (\beta_w(p_{T(w)}(x)))_\sigma$ if $\sigma \in \bar{\Delta}$ and $(\beta_w(p_{T(w)}(z)))_\sigma = (\beta_w(p_{T(w)}(y)))_\sigma$ if $\sigma \in \Sigma \setminus \bar{\Delta}$. Now let $\tilde{x}, \tilde{y} \in S$ such that $\beta_w(p_{T(w)}(x)) = p_\Sigma(\tilde{x})$ and $\beta_w(p_{T(w)}(y)) = p_\Sigma(\tilde{y})$. Define $\tilde{z}$ by $\tilde{z}_\lambda = \tilde{x}_\lambda$ if $\lambda \in \bar{\Delta}$ and $\tilde{z}_\lambda = \tilde{y}_\lambda$ otherwise. Then clearly $\tilde{z} \in S$, and we have $\beta_w(p_{T(w)}(z)) = p_\Sigma(\tilde{z}) \in p_\Sigma(S)$, hence $\bar{S}$ is systematic.

Let $H = \bar{S} - \mathrm{WR}_{(\sigma, y) \in \bar{\Sigma}}\, G_{(\sigma, y)}$, and define $\varphi \colon D(\Sigma) \to H$ as follows. If $g \in D(\Sigma)$, $x \in \bar{S}$, $(\sigma, y) \in \bar{\Sigma}$ then $(x(g\varphi))_{(\sigma, y)} = x_{(\sigma, y)} g_{\sigma, z}$ where $z \in S$ is such that $p_{F(\sigma)}(z) = y$ and $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x))$. We are going to prove that φ is the desired isomorphism.

First of all, we remark that $(g\varphi) \colon \bar{S} \to \bar{X}$ is a well-defined mapping. For this, note that such an element $z \in S$ exists, and that the definition is independent of the choice of z. Namely, if z' is another element with the same properties then $z_\lambda = z'_\lambda$ for all $\lambda > \sigma$, and hence $g_{\sigma, z} = g_{\sigma, z'}$.

Next, we want to show that $(g\varphi)(\bar{S}) \subseteq \bar{S}$. Let $x \in \bar{S}$, and let $w \in p_{\Lambda \setminus \Sigma}(S)$. We have to prove that $\beta_w(p_{T(w)}(x(g\varphi))) \in p_\Sigma(S)$. Define $u \in S$ by $p_{\Lambda \setminus \Sigma}(u) = w$ and $p_\Sigma(u) = \beta_w(p_{T(w)}(x))$. We claim that $\beta_w(p_{T(w)}(x(g\varphi))) = p_\Sigma(ug)$. Let $(\sigma, y) \in T(w)$. Then note that u has the property that $p_{F(\sigma)}(u) = y$ and as $p_{\Lambda \setminus \Sigma}(u) = w$, we also have $p_\Sigma(u) = \beta_{p_{\Lambda \setminus \Sigma}(u)}(p_{T(p_{\Lambda \setminus \Sigma}(u))}(x))$, hence we get $(x(g\varphi))_{(\sigma, y)} = x_{(\sigma, y)} g_{\sigma, u}$. On the other hand, we have $(ug)_\sigma = u_\sigma g_{\sigma, u} = x_{(\sigma, y)} x g_{\sigma, u}$, which establishes the claim.

We are now going to show that $g\varphi$ is bijective. For this, it is enough to prove that $(g\varphi)(g^{-1}\varphi)$ is the identity on $\bar{S}$. We recall that if $z \in S$, $\lambda \in \Lambda$ then $(g^{-1})_{\lambda, x} = (g_{\lambda, xg^{-1}})^{-1}$. Now let $x \in \bar{S}$, and let $\bar{x} = x(g\varphi)$. Then if $(\sigma, y) \in \bar{\Sigma}$ and $z \in S$ is such that $p_{F(\sigma)}(z) = y$ and $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x))$ then we have $\bar{x}_{(\sigma, y)} = x_{(\sigma, y)} g_{\sigma, z}$. Now consider $\bar{z} = zg \in S$. Note that $\bar{z}_\lambda = z_\lambda$ if $\lambda \notin \Sigma$, and hence $p_{F(\sigma)}(\bar{z}) = y$. Also note that $p_\Sigma(\bar{z}) = \beta_{p_{\Lambda \setminus \Sigma}(\bar{z})}(p_{T(p_{\Lambda \setminus \Sigma}(\bar{z}))}(\bar{x}))$. Hence we get $(\bar{x}(g^{-1}\varphi))_{(\sigma, y)} = \bar{x}_{(\sigma, y)}(g^{-1})_{\sigma, \bar{z}}$. Therefore $(x(g\varphi)(g^{-1}\varphi))_{(\sigma, y)} = (\bar{x}(g^{-1}\varphi))_{(\sigma, y)} = \bar{x}_{(\sigma, y)}(g^{-1})_{\sigma, \bar{z}} = \bar{x}_{(\sigma, y)}(g_{\sigma, \bar{z}g^{-1}})^{-1} = (x_{(\sigma, y)} g_{\sigma, z})(g_{\sigma, zgg^{-1}})^{-1} = x_{(\sigma, y)}$. Hence $(g\varphi)(g^{-1}\varphi)$ is the identity on $\bar{S}$.

In the same way, we can show that $(gh)\varphi = (g\varphi)(h\varphi)$, hence φ is a homomorphism $\varphi \colon D(\Sigma) \to \mathrm{Sym}(\bar{S})$. We then have to prove that $g\varphi \in H$. Let $x, x' \in \bar{S}$ with $x\, e(\sigma, y)\, x'$. Then we have $x_{(\sigma_1, y_1)} = x'_{(\sigma_1, y_1)}$ whenever $(\sigma_1, y_1) > (\sigma, y)$. Let $z \in S$ be such that $p_{F(\sigma)}(z) = y$ and $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x))$. We then also have $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x'))$, hence we get $(x'(g\varphi))_{(\sigma, y)} = x'_{(\sigma, y)} g_{\sigma, z}$. From this it follows easily that $g\varphi \in H$.

It now remains to prove that φ is bijective. We first want to show that it is injective, that is, if $1\neq g\in D(\Sigma)$ then $g\varphi\neq 1$. Let $1\neq g\in D(\Sigma)$. Then there exists $\tilde{x}\in S$ such that $\tilde{x}g\neq\tilde{x}$, hence there exists $\sigma\in\Sigma$ such that $(\tilde{x}g)_\sigma\neq\tilde{x}_\sigma$, that is, $\tilde{x}_\sigma g_{\sigma,\tilde{x}}\neq$ $\neq\tilde{x}_\sigma$. Define $x\in\bar{S}$ by $x_{(\bar{\sigma},y)}=\tilde{x}_{\bar{\sigma}}$ for all $(\bar{\sigma},y)\in\bar{\Sigma}$. As S is systematic we have $x\in\bar{S}$. Let $y=p_{F(\sigma)}(\tilde{x})$. We then have $(x(g\varphi))_{(\sigma,y)}=x_{(\sigma,y)}g_{\sigma,\tilde{x}}\neq\tilde{x}_\sigma=x_{(\sigma,y)}$, as we have $p_{F(\sigma)}(\tilde{x})=y$ and also $p_\Sigma(\tilde{x})=\beta_{p_{A\setminus\Sigma}(\tilde{x})}(p_{T(p_{A\setminus\Sigma}(\tilde{x}))}(x))$. Therefore $g\varphi\neq 1$, and hence φ is injective.

Finally, we show that φ is surjective. Let $h\in H$. Then for $(\sigma,y)\in\bar{\Sigma}$, $\bar{x}\in\bar{S}$, we have $h_{(\sigma,y),\bar{x}}$ such that $(x''h)_{(\sigma,y)}=x''_{(\sigma,y)}h_{(\sigma,y),\bar{x}}$ for all $x''\in\bar{S}$ such that $x''\,e(\sigma,y)\,\bar{x}$. We define $g\in D(\Sigma)$ in the following way. If $x\in S$ then $(xg)_\lambda=x_\lambda$ whenever $\lambda\notin\Sigma$, and $(xg)_\lambda=x_\lambda h_{(\lambda,y),q}$ where $y=p_{F(\lambda)}(x)$ and $q\in\bar{S}$ is defined by $q_{(\sigma,y)}=x_\sigma$ for all $(\sigma,y)\in\bar{\Sigma}$ if $\lambda\in\Sigma$. Using the same techniques as above, it follows that $g\in D(\Sigma)$, and that $g\varphi=h$. This proves the theorem.

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Ein neuer elementarer Beweis der Kreisaxiome der hyperbolischen Geometrie

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Unter Kreisaxiomen verstehen wir die folgenden beiden Axiome der elementaren ebenen Geometrie:

K 1. *Wenn A, B, C nicht in einer Geraden gelegene Punkte sind und D ein Punkt der Geraden AB ist, der zwischen A und B liegt, so gibt es einen Punkt B' der Geraden CD, so daß $AB \equiv AB'$ ist.*[1])

K 2. *Es seien A, B, C Punkte auf der Geraden a und A, B', C' Punkte auf derselben oder einer anderen Geraden a', so daß C zwischen A und B liegt und B' zwischen A und C' und $AB \equiv AB'$ ist; wenn dann D ein Punkt ist, so daß $CD \equiv DC'$ wird, so gibt es stets einen Punkt E, so daß $AB \equiv AE$ und $CD \equiv DE$ ist.*[2])

Es ist bekannt, daß sich das erste der obigen beiden Axiome auf Grund der ebenen Axiome der Verknüpfung, der Anordnung und der Kongruenz aus dem zweiten ableiten läßt. (S. z. B. [1], S. 135.) Daß es auch umgekehrt, das zweite auf Grund derselben Axiome aus dem ersten folgt, habe ich in einer früheren Arbeit [7] bewiesen.

Schon F. SCHUR hat mit projektiven Methoden bewiesen [5], daß das Axiom K 1 eine Folge der Axiome I—IV ist, die HILBERT zur Begründung der Bolyai—Lobatschefskyschen Geometrie angenommen hat [3]; außerdem haben J. C. H. GERRETSEN [2] und P. SZÁSZ [8] mittels der auf die „Endrechnung" von HILBERT gegründeten Trigonometrie bzw. analytischen Geometrie bewiesen, daß auch das Axiom K 2 eine Folge der erwähnten Axiome ist. Später habe auch ich auf Grund der Axiome I—IV einen rein elementaren unmittelbaren Beweis beider Axiome

[1]) Wenn ein Punkt einer Geraden im Inneren eines Kreises liegt, so hat die Gerade mit dem Kreis einen Punkt gemein.

[2]) Wenn ein Kreis einen Punkt im Inneren und einen Punkt im Äußeren eines anderen Kreises hat, so haben die beiden Kreise einen Punkt gemein.

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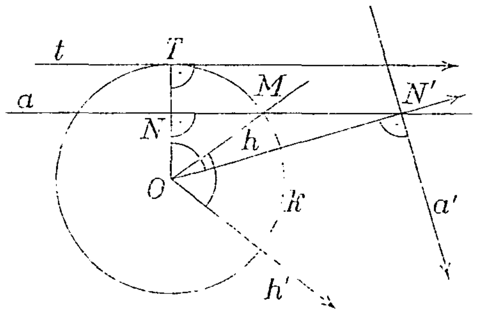
gegeben [6], der auf einer gewissen Zuordnung zwischen den rechtwinkligen Dreiecken und den Vierecken mit drei rechten Winkeln beruht, die schon Lobatschefsky gefunden hat.

In dem nachfolgenden Beweis stützen wir uns nur auf die Theorie der Hjelmslevschen Halbdrehungen, welche auf Grund der Axiome I—III allein abgeleitet werden kann. (S. z. B. [4].)

Zum Beweise sei a eine beliebige Gerade, die den Radius OT des Kreises k um O in einem inneren Punkt N rechtwinklig schneidet (s. nebenstehende Figur), und ferner t in dem Punkt T an den Kreis k gelegte Tangente. Die von O aus nach



einer Richtung hin parallel zu t gezogene Halbgerade schneide a in N'. Wir bezeichnen die Gerade, welche in N' auf ON' senkrecht steht, mit a' und ziehen dann von O aus der T in bezug auf ON' gegenüberliegenden Seite die zu a' parallele Halbgerade h'. Endlich sei h diejenige Halbgerade von O aus, welche auf derselben Seite von OT wie der Punkt N' liegt und überdies mit h' einen Winkel einschließt, der gleich dem von den Halbgeraden ON und ON' gebildeten Winkel wird.

Es ist nun leicht zu sehen, daß bei einer beliebigen Halbdrehung um O dem gemeinsamen uneigentlichen Punkt zweier beliebigen, zueinander parallelen Geraden ein Punkt des Kreises um O entspricht, dessen Radius gleich dem zum Winkel der Halbdrehung als Parallelwinkel gehörigen Lot ist.[3])

Da aber die zum Winkel $(h', OM)=(N'ON)$ als Parallelwinkel gehörende Lotstrecke gleich OT ist, ist $OM=OT$, und daher schneidet die Gerade a den Kreis k in M. Damit ist der Beweis für das Axiom K 1 vollständig erbracht.

Aber auf Grund der Axiome I—III ist das Axiom K 2, wie bereits erwähnt, eine Folge des Axioms K 1.

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[3]) Die Existenz der zu dem Parallelwinkel $\Pi(p)<\frac{\pi}{2}$ gehörigen Lotstrecke p hat schon HILBERT ohne Stetigkeitsbetrachtungen bewiesen. (Vgl. [3], S. 142—144.)

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A characterization of the Radon transform and its dual on Euclidean space

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A. HERTLE [3] gave a characterization of the Radon transform by investigating its behaviour under rotations, dilations and translations and making some restrictions on its range. In this paper we present an alternative characterization of the Radon transform R, for which we do not need any restriction on its range. We also consider the boomerang transform B [5], which is essentially the dual of the Radon transform.

By definition

$$R(f)_{(\omega,p)} = \int_{\langle x,\omega\rangle = p} f(x)\,\mathrm{d}x, \quad B(h)_{(x)} = \int_{S_x^{n-1}} h(\langle \omega, x\rangle \omega)\,\mathrm{d}\omega,$$

where $\omega \in S^{n-1}$, $0 \leqq p \in \mathbf{R}$, $\mathrm{d}x$ and $\mathrm{d}\omega$ is the surface measure, h and f are good enough functions on $\mathbf{R}^n$ (for example continuous functions with compact support) and

$$S_x^{n-1} = \{\omega \in S^{n-1}\colon \langle \omega, x\rangle \geqq 0\}.$$

One can easily verify (see [4]), that the boomerang transform has the following properties:

(1) $B(a_1 f_1 + a_2 f_2) = a_1 B(f_1) + a_2 B(f_2)$

(2) $B(f \circ U) = B(f) \circ U$

(3) $B(f(\delta x))_{(y)} = B(f)_{(\delta y)}$

(4) $B(f)_{(y+z)} = B\left(f\left(x\dfrac{\langle x, x+z\rangle}{\langle x, x\rangle}\right)\right)_{(y)}$

(5) If $f_k \to f$ locally uniformly on $\mathbf{R}^n$ then $B(f_k)_{(x)} \to B(f)_{(x)}$ for any $x \in \mathbf{R}^n$,

where $a_1, a_2 \in \mathbf{R}, f, f_1, f_2, f_k$ are functions on $\mathbf{R}^n$, U is an orthogonal transformation, $0 < \delta \in \mathbf{R}$ and $x, y \in \mathbf{R}^n$.

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Theorem 1. *If a nonzero function-transformation β has properties* (1)—(5) *then* $\beta=\frac{\beta(1)}{B(1)}B$ *on* $C(\mathbf{R}^n\setminus\{0\})$.

Proof. Let $f_i(x)=|x|^i$. The following simple equations prove, that $\beta(f_i)=$ $=f_i\cdot c_i'$, where the constants c_i' depends only on $i\in\mathbf{N}$.

$$\beta(f_i)_{(\delta x)} = \beta(f_i(\delta y))_{(x)} = \delta^i\beta(f_i)_{(x)}$$

$$\beta(f_i)_{(Ux)} = \beta(f_i(Uy))_{(x)} = \beta(f_i)_{(x)}.$$

This implies that if i is even

$$c_i'(t+|x|)^i = \beta(f_i)_{(x+(tx/|x|))} = \beta(f_i(y+te_y\langle e_y, e_x\rangle))_{(x)} =$$

$$= \sum_{j=0}^{i}\binom{i}{j} t^{i-j}\beta(|y|^j\langle e_x, e_y\rangle^{i-j})_{(x)},$$

where $e_x=\frac{x}{|x|}$ and $t\in\mathbf{R}$. Since this is an equality of two polynomials, we have obtained for even i that

$$(*)\qquad c_j'f_j(x) = \beta(|y|^j\langle e_x, e_y\rangle^{i-j})_{(x)}.$$

Now this equation and the condition (4) imply

$$\beta(\langle e, e_y\rangle^i)_{(e_0)} = \beta(\langle e, e_y\rangle^i)_{(e+e_0)} = c_i',$$

where e, e_0 are unit vectors. Let $c_i=B(f_i)/f_i$. The integration of our last equation over $S_{e_0}^{n-1}$ gives

$$c_i'B(1) = \int_{S_{e_0}^{n-1}}\beta(\langle e, e_y\rangle^i)_{(e_0)}\,de = \beta\Big(\int_{S_{e_0}^{n-1}}\langle e, e_y\rangle^i de\Big)_{(e_0)} = \beta(c_i)_{(e_0)} = c_i\beta(1).$$

This way, we have derived the following equation for any even i:

$$B(f_i) = \frac{\beta(1)}{B(1)}B(f_i).$$

Furthermore if i is even

$$\beta(f_i(x+y)) = \beta(\langle x+y, x+y\rangle^{i+2})$$

$$= \sum_{k,l,m}\binom{i/2}{k,\,l,\,m}2^m|x|^{2l+m}\beta(|y|^{2k+m}\langle e_x, e_y\rangle^m),$$

where $k+l+m=i/2$ and $k,l,m\in\mathbf{N}$. Taking into account $(*)$ this gives rise to (by calculating $B(f_i(x+y))$)

$$\beta(f_i(x+y)) = \frac{\beta(1)}{B(1)}B(f_i(x+y)).$$

Since the translates of radial polynomials are locally uniformly dense in $C(\mathbf{R}^n)$ this completes the proof.

Theorem 2. *If a nonzero function-transformation ϱ has the following properties then $\varrho=cR$ on $C_c(\mathbf{R}^n)$, the compactly supported continuous functions, for some $c\in\mathbf{R}$.*

(1′) $\varrho(a_1 f_1+a_2 f_2) = a_1\varrho(f_1)+a_2\varrho(f_2)$

(2′) $\varrho(f\circ U) = \varrho(f)\circ U$

(3′) $\varrho(f(\delta x))_{(\omega,r)} = \delta^{1-n}\varrho(f)_{(\omega,\delta r)}$

(4′) $\varrho(f(x+y))_{(\omega,r)} = \varrho(f)_{(\omega,r+\langle\omega,y\rangle)}$

(5′) *If $f_k\to f$ locally uniformly on $\mathbf{R}^n$ then $\varrho(f_k)\to\varrho(f)$ (with common support).*

The notations are the same as in Theorem 1.

Proof. Let $\hat{f}=B(\varrho(f))$. Then we have

(a) $(a_1 f_1+a_2 f_2)\hat{} = a_1\hat{f}_1+a_2\hat{f}_2$

(b) $(f\circ U)\hat{} = \hat{f}\circ U$

(c) $(f(\delta x))\hat{}(y) = \delta^{1-n}\hat{f}(\delta y)$

(d) $(f(x+y))\hat{}(z) = \hat{f}(z+y)$

(e) If $f_k\to f$ locally uniformly on $\mathbf{R}^n$ then $\hat{f}_k\to\hat{f}$ (with common support).

It is well known [1], that if a transform satisfies (a), (d) and (e) then there is a function g on $\mathbf{R}^n$ such that

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(y)\, g(x-y)\, \mathrm{d}y.$$

A short calculation with (b) and (c) gives that

$$\hat{f}(x) = \int_{\mathbf{R}^n} f(y)\frac{d}{|x-y|}\,\mathrm{d}y,$$

where d is a suitable constant. This implies by [2, p. 104] that $\hat{f}(x)=cB(R(f))$ for some constant c. Thus we have for any function f that

$$B(\varrho(f)-cR(f)) = 0.$$

From Theorem 1.3 of [4] we know the invertibility of the boomerang transform on the set of radial continuous functions. So if we prove that the continuity of f implies the continuity of $\varrho(f)$ then we shall obtain that ϱ is equal to cR on the continuous radial functions. Since the translates of radial continuous functions are locally uniformly dense in $C_c(\mathbf{R}^n)$ this will imply the theorem.

Thus to finish the proof we only have to prove the continuity of $\varrho(f)$ for any continuous radial function f. For any $\varepsilon>0$ we have

$$\varrho(f)_{(r+\varepsilon)}-\varrho(f)_{(r)}=\varrho\left(\left[\frac{r+\varepsilon}{r}\right]^{n-1}f\left(\frac{r+\varepsilon}{r}x\right)\right)_{(r)}-\varrho(f)_{(r)}=$$

$$=\varrho\left(\left[\frac{r+\varepsilon}{r}\right]^{n-1}f\left(\frac{r+\varepsilon}{r}x\right)-f(x)\right)_{(r)}$$

If ε tends to zero then

$$\left[\frac{r+\varepsilon}{r}\right]^{n-1}f\left(\frac{r+\varepsilon}{r}x\right)-f(x)\to 0$$

uniformly on $\operatorname{supp}(f)$. This just means that $\varrho(f)$ is a continuous function.

Theorem 1 (resp. Theorem 2) is valid for any other function space in which $C(\mathbf{R}^n\setminus\{0\})$ (resp. $C_c(\mathbf{R}^n)$) is dense.

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Представление абсолютно непрерывных функций многих переменных

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Для абсолютно непрерывных функций одной переменной $f(x)$ хорошо известна формула

$$f(x)-f(0)=\int_0^x f'(t)\,dt, \tag{1}$$

где $f'(t)$ существует почти всюду, а x любая точка из отрезка $[0, 1]$, на котором определена $f(x)$. Цель настоящей статьи-получить аналог представления (1) для абсолютно непрерывных функций n действительных переменных.

Условимся в некоторых обозначениях. Через $x=(x_1, \ldots, x_n)$ обозначаются точки пространства $\mathbf{R}^n$, а через $|A|$-Лебеговая мера множества $A\subset\mathbf{R}^n$. Если $\Delta=\prod_{i=1}^n \Delta_i$, где $\Delta_i=[a_i^{(1)}; a_i^{(2)})$ любой полуоткрытый интервал на оси OX_i, то

$$\Delta F=\Delta_n(\Delta_{n-1}\ldots(\Delta_1 F(x))), \tag{2}$$

где

$$\Delta_i g(x)=g(x_1, \ldots, a_i^{(2)}, \ldots, x_n)-g(x_1, \ldots, a_i^{(1)}, \ldots, x_n). \tag{3}$$

Методом математической индукции по количеству переменных n, легко можно убедиться, что

$$\Delta F=\sum_{\substack{i_j=1,2\\ j=1,2,\ldots,n}}(-1)^{i_1+\cdots+i_n}F(a_1^{(i_1)}, \ldots, a_n^{(i_n)}). \tag{4}$$

Определение 1. Скажем, что функция $F(x)$, определенная на $[0, 1]^n$ *абсолютно непрерывна,* если для любого $\varepsilon>0$ существует $\delta>0$ такое, что для любого набора прямоугольников $\Delta^{(1)}, \Delta^{(2)}, \ldots; \Delta^{(k)}=\prod_{i=1}^n \Delta_i^{(k)}, \Delta^{(k)}\subset[0, 1]^n$, удовлетворяющих условиям

$$\Delta^{(k)}\cap\Delta^{(k')}=\emptyset, \quad \text{когда} \quad k\neq k', \tag{5}$$

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и

(6) $$\sum_k |\Delta^{(k)}| < \delta,$$

выполняется

(7) $$\sum_k |\Delta^{(k)} F| < \varepsilon.$$

Определение 2. Функцию $F(k)$, определенную на $[0, 1]^n$, назовем *функцией ограниченной вариации*, если сучествует зависящая от $F(x)$ постоянная M, такая что при любом представлении $[0, 1)^n = \bigcup_k \Delta^{(k)}$, $\Delta^{(k)} \cap \Delta^{(k')} = \emptyset$, когда $k \neq k'$, выполняется

(8) $$\sum_k |\Delta^{(k)} F| \leqq M.$$

Верхнюю грань сумм (8) обозначим $\bigvee_{[0,1)^n} (F)$ и назовем вариацией функции $F(x)$ на $[0, 1)^n$.

Как и для функций одной переменной (см. [1]), легко выдеть, что абсолютно непрерывная функция n переменных $F(x)$ имеет ограниченную вариацию. Хорошо известно, что абсолютно непрерывные функции одной переменной являются так же непрерывными, а функции ограниченной вариации непрерывны всюду, кроме, быть может, некоторого счетного множества точек. В случае многих переменных это уже не так. Абсолютно непрерывная функция многих переменных может не быть даже измеримой. Действительно, если $g(t)$ некоторая неизмеримая функция одной переменной, то функция $F(x_1, \ldots, x_n) = g(x_1)$ будет неизмеримой, и для любого выполнится $\Delta F = 0$.

Как и для функций одной переменной, для функций многих переменных, имеющих ограниченную вариацию, можно доказать возможность представления

(9) $$F(x) = \Phi(x) - \Psi(x),$$

где $\Phi(x)$ и $\Psi(x)$ при любом Δ удовлетворяют условию

(10) $$\Delta\Phi \geqq 0, \quad \Delta\Psi \geqq 0.$$

В представлении (9) в качестве $\Phi(x)$ можно взять

(11) $$\Phi(x_1, \ldots, x_n) = \bigvee_{\bigcap_{i=1}^{n} [0, x_i)} (F)$$

причем, если $F(x)$-абсолютно непрерывная функция, то функции $\Phi(x)$ и $\Psi(x)$ тоже абсолютно непрерывны (в условиях Определения 1). Доказательства этих утверждений полностью совпадают с доказательствами аналогичных фактов для функций одной переменной (см. [1]). Поэтому мы их приводить не будем.

Несмотря на то, что абсолютно непрерывная функция многих переменных может не быть измеримой, имеет место следующая

Теорема. *Для того, чтобы $F(x)$ была абсолютно непрерывной функцией, необходимо и достаточно, чтобы она имела представление*

$$F(x_1, \ldots, x_n) = \sum_j F_j(x_1, \ldots, x_n) + \int_0^{x_n} \ldots \int_0^{x_1} \varphi(t_1, \ldots, t_n)\, dt_1, \ldots, dt_n, \tag{12}$$

где $F_j(x_1, \ldots, x_n)$ — некоторые функции, зависящие от меньшего числа переменных, а $\varphi(t_1, \ldots, t_n)$ — интегрируемая по Лебегу функция.

Отметим, что в случае $n=2$ и при дополнительных условиях, что функции $F(0, x_2)$ и $F(x_1, 0)$ абсолютно неприрывны (как функции от одной переменной), теорема доказана в [2] методами, отличающимися от ниже изложенных.

Доказательство теоремы. *Достаточность.* Нетрудно проверить, что для функций $F_j(x)$ имеет место (см. (4))

$$\Delta F_j = 0 \quad \text{для любого} \quad \Delta. \tag{13}$$

Поетому

$$\Delta F = \Delta \left(\int_0^{x_n} \ldots \int_0^{x_1} \varphi(t_1, \ldots, t_n)\, dt_1, \ldots, dt_n \right) = \int_\Delta \varphi(t)\, dt. \tag{14}$$

Из абсолютной непрерывности интеграла Лебега и из (14) получаем, что $F(x)$ абсолютно непрерывна в смысле Определения 1.

Необходимость. Учитывая разложение (9), (10), без ограничения общности, можно считать, что функция $F(x)$ удовлетворяет условию

$$\Delta F \geqq 0 \quad \text{для любого} \quad \Delta. \tag{15}$$

Обозначим

$$\tilde{F}(x) = \Delta_x F, \quad \text{где} \quad \Delta_x = \prod_{i=1}^{n} [0, x_i). \tag{16}$$

Очевидно, что $\tilde{F}(x)$ есть $F(x)$ плюс некоторые функции (см. (4)) меньшего числа переменных. Поэтому

$$\Delta \tilde{F} = \Delta F \geqq 0. \tag{17}$$

Лемма 1. *Если* $\Delta^{(k)}=\prod_{i=1}^{n}\Delta_i^{(k)}$, $k=1,2,\ldots,m$, *удовлетворяют условиям* $\Delta^{(k)}\cap\Delta^{(k')}=\emptyset$, $k\neq k'$, *и* $\bigcup_{k=1}^{m}\Delta^{(k)}=\Delta=\prod_{i=1}^{n}\Delta_i$, *то*

$$\Delta F=\sum_{r=1}^{m}\Delta^{(k)}F. \tag{18}$$

Действительно, когда $m=2$, то при некотором i_0, $\Delta_{i_0}=\Delta_{i_0}^{(1)}\cup\Delta_{i_0}^{(2)}$, $\Delta_{i_0}^{(1)}\cap\cap\Delta_{i_0}^{(2)}=\emptyset$, $\Delta_i=\Delta_i^{(1)}=\Delta_i^{(2)}$, при $i\neq i_0$, и тогда легко выдеть, что в сумме $\sum_{k=1}^{2}\Delta^{(k)}F$ (см. (4)) остаются только те слагаемые, которые нужны для представления ΔF (по формуле (4)).

Случай, когда $\Delta_i=\bigcup_{j=1}^{m_i}\Delta_i^{(j)}$, $\Delta_i^{(j)}\cap\Delta_i^{(j')}=\emptyset$, $j\neq j'$ и

$$\{\Delta^k\}=\left\{\Delta'\colon \Delta'=\prod_{i=1}^{n}\Delta_i^{(j_i)},\ 1\leqq j_i\leqq m_i\right\} \tag{19}$$

т. е. когда каждая компонента Δ_i прямоугольника Δ разбыта на непересекающиеся интервалы $\Delta_i^{(j)}$, а прямоугольники $\Delta^{(k)}$ являются всевозможными декартовыми произведениями этих $\Delta_i^{(j)}$, легко приводится к случаю $m=2$.

Общий случай приводится к случаю (19) путем размельчания прямоугольников $\Delta^{(k)}$. Лемма доказана.

Из (17), (18) следует, что если для каждого прямоугольника определить

$$\nu(\Delta)=\Delta F, \tag{20}$$

то $\nu(\Delta)$ будет мерой, определенной на полукольце прямоугольников $\{\Delta\}$ (см. [3], гл. V, § 2). Эта мера единственным образом продолжается на минимальное кольцо, содержащее полукольцо $\{\Delta\}$.

Из того, что $\nu(\Delta)$ является мерой, легко выводится непрерывность функции $\tilde{F}(x)$. Действительно,

$$|\tilde{F}(x^{(1)})-\tilde{F}(x^{(2)})|=|\Delta_{x^{(1)}}F-\Delta_{x^{(2)}}F|= \tag{21}$$
$$=|\nu(\Delta_{x^{(1)}})-\nu(\Delta_{x^{(2)}})|\leqq\nu\big((\Delta_{x^{(1)}}\setminus\Delta_{x^{(2)}})\cup(\Delta_{x^{(2)}}\setminus\Delta_{x^{(1)}})\big).$$

Учитывая, что $\Delta_{x^{(1)}}\setminus\Delta_{x^{(2)}}$ и $\Delta_{x^{(2)}}\setminus\Delta_{x^{(1)}}$ являются объединением конечнего числа прямоугольников, сумма Лебеговских мер которых стремится к нулю, когда $\|x^{(1)}-x^{(2)}\|\to 0$, из (21) и Определения 1 получаем непрерывность функции $\tilde{F}(x)$.

Лемма 2. *Мера* ν σ*-аддитивная мера на* $\{\Delta\}$.

Для доказательства леммы достаточно убедиться, что если $\Delta=\bigcup\limits_{k=1}^{\infty}\Delta^{(k)}$ то (см. [3], гл. V, § 2)

$$\nu(\Delta)\leqq\sum_{k=1}^{\infty}\nu(\Delta^{(k)}). \tag{22}$$

Из непрерывности функции $\tilde{F}(x)$ и (4) следует, что для произвольного $\varepsilon>0$ найдутся Δ', $\Delta^{(k)'}$, $k=1,2,\ldots$, такие что (см. также (17), (20))

$$\bar{\Delta}'\subset\Delta,\quad \nu(\Delta')\geqq\nu(\Delta)-\varepsilon, \tag{23}$$

$$\Delta^{(k)}\subset\operatorname{int}(\Delta^{(k)'}),\quad \nu(\Delta^{(k)'})\leqq\nu(\Delta^{(k)})+\varepsilon/2^k,\quad k=1,2,\ldots. \tag{24}$$

Очевидно, что замкнутое множество $\bar{\Delta}'$ покрывается открытыми прямоугольниками $\operatorname{int}(\Delta^{(k)'})$. Следовательно существует конечный набор $\{\operatorname{int}(\Delta^{(k_i)'})\}$, который тоже покрывает $\bar{\Delta}'$. Тогда $\Delta'\subset\bigcup\limits_{i}\Delta^{(k_i)'}$, и из (23), (24) получаем

$$\nu(\Delta)\leqq\nu(\Delta')+\varepsilon<\sum_{i}\nu(\Delta^{(k_i)'})+\varepsilon\leqq\sum_{k=1}^{\infty}\nu(\Delta^{(k)'})+\varepsilon\leqq\sum_{k=1}^{\infty}\nu(\Delta^{(k)})+2\varepsilon. \tag{25}$$

Из (25) следует (22), а из (22), как уже было отмечено, следует утверждение леммы.

Из Леммы 2 следует (см. [3]), что меру ν можно продолжить на σ-алгебру множеств, измеримых по Лебегу, и эта мера будет σ-аддитивной. Из Определения 1 следует, что мера ν абсолютно непрерывна относительно меры Лебега, т. е. если $|A|=0$, то $\nu(A)=0$. В силу теоремы Радона—Никодима (см. [3], гл. VI, § 5) существует интегрируемая (по Лебегу) функция $\varphi(t_1,\ldots,t_n)$, такая что для любого измеримого по Лебегу множества A имеет место

$$\nu(A)=\int_A\varphi(t)\,dt. \tag{26}$$

В частном случае, когда $A=\prod\limits_{i=1}^{n}[0,x_i]$, из (20), (26), (17) получается

$$\tilde{F}(x)=\int_0^{x_n}\ldots\int_0^{x_1}\varphi(t_1,\ldots,t_n)\,dt_1\ldots dt_n. \tag{27}$$

Учитывая, что $\tilde{F}(x)$ является суммой $F(x)$ и некоторых функций меньшего числа переменных (см. (16) и (4)), из (27) получаем (12). Теорема доказана.

Полученное соотношение (27) эквивалентно следующему (см. (16))

$$\Delta_x F=\int_0^{x_n}\ldots\int_0^{x_1}\varphi(t_1,\ldots,t_n)\,dt_1\ldots dt_n \tag{28}$$

которое является аналогом равенства (1) для функций многих переменных.

Существенной разницей между формулами (1) и (28) является то, что в (1) известны способы нахождения подинтегральной функции, а в (28) лишь утверждается существование функции $\varphi(t)$. С целю устронения этой разницы рассмотрим следующие функции. Через $G^{(k)}$ обозначим n-мерные кубы вида $[i_1/2^k, (i_1+1)/2^k)\times\ldots\times[i_n/2^k, (i_n+1)/2^k)$, где $0\leqq i_n<2^k$. Для $k=0, 1, 2, \ldots$ положим

$$\Phi_k(x) = \frac{1}{|G^{(k)}|}\int_{G^{(k)}}\varphi(t)\,dt, \quad \text{когда} \quad x\in G^{(k)}. \tag{29}$$

Из (26), (20) получаем

$$\Phi_k(x) = \frac{G^{(k)}F}{|G^{(k)}|}, \quad \text{когда} \quad x\in G,^{(k)} \quad k = 0, 1, 2, \ldots. \tag{30}$$

С другой стороны, из известной теоремы Йессена, Марцинкевича и Зигмунда (см. [4], теорема 6) следует, что

$$\lim_{k\to\infty}\Phi_k(x) = \varphi(x) \quad \text{п.в. на} \quad [0, 1]^n. \tag{31}$$

Итак, функция $\varphi(x)$ есть предел последовательности функций $\Phi_k(x)$ которые определяются функцией $F(x)$ формулами (30).

Замечание. Теорему можно доказать также, исходя из функций $\Phi_k(x)$ определяемих формулами (30). Укажем шаги такого способа доказательства. Во-первых, можно убедится, что $\{\Phi_k(x)\}_{k=0}^{\infty}$ является равностепенно абсолютно непрерывно интегрируемой мартингальной последовательностью, и поэтому последовательность $\{\Phi_k(x)\}_{k=0}^{\infty}$ почти всюду и в метрике L_1 сходится к некоторой функции $\varphi(x)$. Далее, легко проверяется, что для любого $x=(x_1, \ldots, x_n)=$ $=(i_1/2^{k_1}, \ldots, i_n/2^{k_n})$, начиная с некоторого k_0 выполняется

$$\Delta_x F = \int_0^{x_n}\ldots\int_0^{x_1}\Phi_k(t_1, \ldots, t_n)\,dt_1\ldots dt_n. \tag{32}$$

Следовательно

$$\Delta_x F = \int_0^{x_n}\ldots\int_0^{x_1}\varphi(t_1, \ldots, t_n)\,dt_1\ldots dt_n \quad \text{при любом} \quad x_j = i_j/2^{k_j},\ j = 1, \ldots, n. \tag{33}$$

Наконец, так как обе стороны равенства (33) являются непрерывными функциями и точки $(i_1/2^{k_1}, \ldots, i_n/2^{k_n})$ всюду плотны, то (33) выполняется при любом $(x_1, \ldots, x_n)$.

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Further sharpening of inequalities of Hardy and Littlewood

L. LEINDLER

In a previous paper [2] we generalized some classical and very useful inequalities of HARDY and LITTLEWOOD [1]. One special case of our results states the following inequalities:

If $\lambda_n>0$ *and* $a_n\geqq 0$ $(n=1,2,\ldots)$ *then we have*

$$\left.\begin{array}{ll}(1) & \sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=1}^{n}a_k\Big)^p \leqq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=n}^{\infty}\lambda_k\Big)^p \\ (2) & \sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=n}^{\infty}a_k\Big)^p \leqq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=1}^{n}\lambda_k\Big)^p\end{array}\right\}\quad \textit{for}\quad p\geqq 1,$$

and

$$\left.\begin{array}{ll}(3) & \sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=n}^{\infty}\lambda_k\Big)^p \leqq 8\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=1}^{n}a_k\Big)^p \\ (4) & \sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=1}^{n}\lambda_k\Big)^p \leqq 9\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=n}^{\infty}a_k\Big)^p\end{array}\right\}\quad \textit{for}\quad 0<p\leqq 1.$$

These inequalities reduce to those of Hardy and Littlewood if $\lambda_n=n^{-c}$ with $c>1$ in (1) and (3); and with $c\leqq 1$ in (2) and (4).

The factor p^p is best possible one, but 8 and 9 are not. In the present note we improve, among others, inequalities (3) and (4) proving that the constants 8 and 9 can be replaced by p^{-p} and this is the best possible one. It is easy to see that $p^{-p}\leqq\leqq e^{1/e}<1{,}45$ holds for any $0<p\leqq 1$. Having these improved inequalities (3) and (4), we can state that inequalities (1) and (2) hold for $p\geqq 1$ and their reversed ones hold for $0<p\leqq 1$.

In [2] it was assumed only that $\lambda_n\geqq 0$ and in this more general form we proved the following theorem.

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Theorem A. *Let* $a_n \geqq 0$ *and* $\lambda_n \geqq 0$ $(n=1, 2, \ldots)$ *be given. Let* $\nu_1 < \ldots < \nu_n < \ldots$ *denote the indices for which* $\lambda_{\nu_n} > 0$. *Let* N *denote the number of the positive terms of the sequence* λ_n, *provided this number is finite; in the contrary case set* $N = \infty$. *Set* $\nu_0 = 0$, *and if* $N < \infty$ *then* $\nu_{N+1} = \infty$. *Using the notations*

$$A_{m,n} := \sum_{i=m}^{n} a_i \quad \text{and} \quad \Lambda_{m,n} := \sum_{i=m}^{n} \lambda_i \quad (1 \leqq m \leqq n \leqq \infty),$$

we have the following inequalities:

$$\left.\begin{aligned} &(1') && \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \\ &(2') && \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \leqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p \end{aligned}\right\} \text{ for } p \geqq 1$$

(the constant p^p *being the best possible one) and*

$$\left.\begin{aligned} &(3') && \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \leqq 8 \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \\ &(4') && \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p \leqq 9 \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \end{aligned}\right\} \text{ for } 0 < p \leqq 1.$$

The aim of this note is to prove

Theorem. *Under the assumptions of Theorem* A *the opposite inequalities of* (1′) *and* (2′) *hold for* $0 < p \leqq 1$, *and the constant* p^p *is best possible one, in this case, too.*

Theorem A and Theorem imply immediately

Corollary 1. *If* $\lambda_n > 0$ *and* $a_n \geqq 0$ *then* (1) *and* (2) *hold for* $p \geqq 1$, *and their opposite inequalities for* $0 < p \leqq 1$.

If $0 < p \leqq 1$ then we can reduce the restriction $\lambda_n > 0$ of Corollary 1 to $\lambda_n \geqq 0$, i.e. we can prove

Corollary 2. *For any* $a_n \geqq 0$ *and* $\lambda_n \geqq 0$, *if* $0 < p \leqq 1$, *we have*

$$\sum_{n=1}^{\infty} \lambda_n^{1-p} \Big(\sum_{k=n}^{\infty} \lambda_k\Big)^p a_n^p \leqq p^{-p} \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=1}^{n} a_k\Big)^p \tag{3''}$$

and

$$\sum_{n=1}^{\infty} \lambda_n^{1-p} \Big(\sum_{k=1}^{n} \lambda_k\Big)^p a_n^p \leqq p^{-p} \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=n}^{\infty} a_k\Big)^p. \tag{4''}$$

This is an immediate consequence of the facts that

$$\lambda_{\nu_n}^{1-p} A_{\nu_n,\infty}^p \Big(\sum_{i=\nu_{n-1}+1}^{\nu_n} a_i\Big)^p \geqq \sum_{i=\nu_{n-1}+1}^{\nu_n} \lambda_i^{1-p} A_{i,\infty}^p a_i^p \tag{5}$$

and

$$\lambda_{\nu_n}^{1-p} A_{1,\nu_n}^p \Big(\sum_{i=\nu_n}^{\nu_{n+1}-1} a_i\Big)^p \geqq \sum_{i=\nu_n}^{\nu_{n+1}-1} \lambda_i^{1-p} A_{1,i}^p a_i^p \tag{6}$$

obviously hold — since $\lambda_i=0$ if $i\neq\nu_n$ — so by the opposite inequalities of (1′) and (2′) to be proved in Theorem, regarding inequalities (5) and (6), we get (3″) and (4″).

Proofs

1. First we prove the opposite of (1′) for $0<p\leqq 1$, i.e.

$$\Sigma_1 := \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \geqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} A_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p =: \Sigma_2. \tag{1.1}$$

We may assume that Σ_1 has a positive finite value. For $p=1$ (1.1) is obvious, we have only to interchange the order of the summations. If $0<p<1$ then we set the following notations:

$$\alpha_n := A_{\nu_{n-1}+1,\nu_n},\quad \beta_0 = 0,\quad \beta_n := \sum_{k=1}^{n} \alpha_k,\quad \varrho_n := \lambda_{\nu_n},$$

and

$$R_n := A_{\nu_n,\infty}$$

for every $1\leqq n\leqq N$. If $N<\infty$ then let

$$R_{N+1} := \varrho_{N+1} := 0.$$

Then, for any positive integer $m(\leqq N)$, we have

$$\sum_{k=1}^{\nu_m} \lambda_k A_{1,k}^p = \sum_{n=1}^{m} \varrho_n \beta_n^p = \sum_{n=1}^{m} (R_n - R_{n+1})\beta_n^p = \sum_{n=1}^{m} R_n(\beta_n^p - \beta_{n-1}^p) - R_{m+1}\beta_m^p,$$

whence

$$\sum_{n=1}^{m} R_n(\beta_n^p - \beta_{n-1}^p) \leqq R_{m+1}\beta_m^p + \sum_{k=1}^{\nu_m} \lambda_k A_{1,k}^p \leqq \sum_{k=1}^{\infty} \lambda_k A_{1,k}^p = \Sigma_1 < \infty \tag{1.2}$$

follows for any $m\leqq N$.

Let μ be the smallest positive integer having the property $\beta_\mu>0$.

An easy calculation gives that

$$(1.3)\qquad \sum_{n=\mu}^{m} R_n(\beta_n^p-\beta_{n-1}^p) \geqq \sum_{n=\mu}^{m} R_n p\alpha_n \beta_n^{p-1} =$$

$$= p\sum_{n=\mu}^{m} (\varrho_n^{(1/p)-1} R_n\alpha_n)(\varrho_n^{1-(1/p)}\beta_n^{p-1}) =: p\Sigma_3.$$

Now using the classical inequality of Hölder with p and $\dfrac{p}{p-1}$ we get

$$(1.4)\qquad \Sigma_3 \geqq \Big(\sum_{n=\mu}^{m} \varrho_n^{1-p} R_n^p \alpha_n^p\Big)^{1/p}\Big(\sum_{n=\mu}^{m} \varrho_n\beta_n^p\Big)^{1-1/p}.$$

By (1.2), (1.3) and (1.4) we have

$$\sum_{k=1}^{\infty} \lambda_k A_{1,k}^p \geqq p\Big(\sum_{n=1}^{\infty} \varrho_n^{1-p} R_n^p \alpha_n^p\Big)^{1/p}\Big(\sum_{n=1}^{\infty} \varrho_n\beta_n^p\Big)^{1-1/p} =$$

$$= p\Big(\sum_{n=1}^{\infty} \lambda_{\nu_n}^{1-p} A_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p\Big)^{1/p}\Big(\sum_{n=1}^{\infty} \lambda_{\nu_n}\Big(\sum_{k=1}^{\nu_n} a_k\Big)^p\Big)^{1-1/p},$$

whence (1.1) obviously follows.

2. Secondly we prove the opposite of (2′) for $0<p\leqq 1$, i.e.

$$(1.5)\qquad \Sigma_4 := \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \geqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} A_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p =: \Sigma_5.$$

As before the case $p=1$ is trivial and we may assume that Σ_4 has a positive finite value. Using the previous notations and let $\alpha_n^* := A_{\nu_n,\nu_{n+1}-1}$, $\gamma_n := \sum_{k=n}^{N} \alpha_k^*$ and $B_n := A_{1,\nu_n} = \sum_{k=1}^{n} \varrho_k$; furthermore let ω denote the greatest natural number, if there exists, for which $A_{\nu_\omega,\infty}>0$, otherwise let $\omega:=\infty$. If $\Omega:=\min(N,\omega)$, then for any $1\leqq n\leqq k\leqq \Omega$ $\gamma_n = A_{\nu_n,\infty}>0$ and we have the estimation

$$(1.6)\qquad \sum_{n=1}^{k+1} \varrho_n\gamma_n^p = \sum_{n=1}^{k+1} (B_n-B_{n-1})\gamma_n^p =$$

$$= \sum_{n=1}^{k} B_n(\gamma_n^p-\gamma_{n+1}^p) + B_{k+1}\gamma_{k+1} \geqq \sum_{n=1}^{k} B_n(\gamma_n^p-\gamma_{n+1}^p).$$

By (1.6) it is clear that for any $k\leqq\Omega$

$$(1.7)\qquad \Sigma_4 \geqq \sum_{n=1}^{k} B_n(\gamma_n^p-\gamma_{n+1}^p) \geqq \sum_{n=1}^{k} B_n p\alpha_n^*\gamma_n^{p-1} =$$

$$= p\sum_{n=1}^{k} (\varrho_n^{1/p-1} B_n\alpha_n^*)(\varrho_n^{1-1/p}\gamma_n^{p-1})$$

holds. Using again the classical inequality of Hölder with p and $\frac{p}{p-1}$ we obtain, by (1.7), that

$$\Sigma_4 \geqq p\Big(\sum_{n=1}^{k} \varrho_n^{1-p} B_n^p (\alpha_n^*)^p\Big)^{1/p} \Big(\sum_{n=1}^{k} \varrho_n \gamma_n^p\Big)^{1-1/p} =$$

$$= p\Big(\sum_{n=1}^{k} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p\Big)^{1/p} \Big(\sum_{n=1}^{k} \lambda_{\nu_n} A_{\nu_n,\infty}^p\Big)^{1-1/p},$$

whence (1.5) follows immediately.

3. To verify that the constant p^p in the opposite inequalities of (1') and (2') for $0<p\leqq 1$ is best possible it is enough to consider e.g. the following special case:

$$a_n := \big(\log(n+1)\big)^{-2/p} \quad \text{and} \quad \lambda_n = n^{-1-p} \quad \text{for} \quad n \geqq N \to \infty$$

and everything is zero for $n<N$.

The proof of Theorem is completed.

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Fourier series with positive coefficients and generalized Lipschitz classes

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1. L. LEINDLER ([3]—[6]) investigated the relations between function classes defined by the rate of strong approximation of functions by Fourier series and the classes determined by the modulus of continuity of the functions.

Following G. G. LORENTZ [7] and R. P. BOAS [2] we shall prove theorems giving coefficient-conditions assuring that a function belong to function classes defined in terms of modulus of continuity by L. Leindler.

Combining these results and those of L. Leindler mentioned above we can get coefficient-conditions for a function to belong to a function class defined by the rate of strong approximation by Fourier series.

2. Before formulating our results we give a couple of definitions, notations and theorems.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

be its Fourier series. Denote by $s_n = s_n(x) = s_n(f; x)$ the n-th partial sum of (1). For any positive β and p L. LEINDLER [3] defined the following strong means and function classes

$$h_n(f; \beta; p) = \left\| \left\{ \frac{1}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} |s_k - f|^p \right\}^{1/p} \right\|$$

$$H(\beta, p, \omega) = \{f\colon h_n(f; \beta; p) = O(\omega(1/n))\},$$

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where $\|\cdot\|$ denotes the usual maximum norm and $\omega(\delta)$ is a modulus of continuity having the following properties

$$\omega(0)=0,\quad \omega(\delta_1+\delta_2)\leqq\omega(\delta_1)+\omega(\delta_2)\quad \text{for any}\quad 0\leqq\delta_1\leqq\delta_2\leqq\delta_1+\delta_2\leqq 2\pi.$$

Furthermore we consider the following function classes

$$H^\omega=\{f\colon \|f(x+h)-f(x)\|=O(\omega(h))\}$$
$$(H^\omega)^*=\{f\colon \|f(x+h)+f(x-h)-2f(x)\|=O(\omega(h))\}.$$

If $\omega(\delta)=\delta^\alpha$ $(0<\alpha\leqq 1)$ then H^{δ^α} are the known Lipschitz classes.

In [3] L. LEINDLER (see also in [6], p. 153) proved, among others, the following result.

$$\left.\begin{array}{lll} (2) & H(\beta,p,\delta^\alpha)\equiv H^{\delta^\alpha} & \text{for}\quad 0<\alpha<1 \\ (3) & H(\beta,p,\delta)\equiv(H^\delta)^* & \text{for}\quad \alpha=1 \end{array}\right\}\quad \text{if}\quad \beta>\alpha p.$$

G. G. LORENTZ [7] proved in 1948 that *if* $\lambda_n\downarrow 0$ *and* λ_n *are the Fourier sine or cosine coefficients of* f *then* $f\in H^{\delta^\alpha}$ $(0<\alpha<1)$ *if and only if* $\lambda_n=O(n^{-1-\alpha})$. This result and some others were generalized by R. P. BOAS [2] in 1967 as follows: *Let* $\lambda_n\geqq 0$ *and let* λ_n *be the Fourier sine or cosine coefficients of* f. *Then* $f\in H^{\delta^\alpha}$ $(0<\alpha<1)$ *if and only if*

$$\sum_{k=n}^{\infty}\lambda_k=O(n^{-\alpha}), \tag{4}$$

or equivalently

$$\sum_{k=1}^{n}k\lambda_k=O(n^{1-\alpha}). \tag{5}$$

Combining this result and the result of L. Leindler mentioned above we have that if $\lambda_n\geqq 0$ and λ_n are sine or cosine coefficients of f then the following three relations are equivalent:

$$f\in H(\beta,p,\delta^\alpha);\quad \sum_{k=n}^{\infty}\lambda_k=O(n^{-\alpha});\quad \sum_{k=1}^{n}k\lambda_k=O(n^{1-\alpha}), \tag{6}$$

if $0<\alpha<1$ and $\beta>\alpha p$.

L. Leindler has extended results (2) and (3) from H^{δ^α} to H^ω at least for certain special but more general class of moduli of continuity than $\omega(\delta)=\delta^\alpha$.

Next we give the definition of this class of moduli of continuity (see [4], [5] and in [6], p. 154). Let $\omega_\alpha(\delta)$ denote the modulus of continuity having the following properties for $0\leqq\alpha\leqq 1$:

i) for any $\alpha'>\alpha$ there exists a natural number $\mu=\mu(\alpha')$ such that

$$2^{\mu\alpha'}\omega_\alpha(2^{-n-\mu})>2\omega_\alpha(2^{-n})\quad \text{holds for all}\quad n(\geqq 1); \tag{7}$$

ii) for every natural number ν there exists a natural number $N(\nu)$ such that

$$2^{\nu\alpha}\omega_\alpha(2^{-n-\nu}) \leqq 2\omega_\alpha(2^{-n}) \quad \text{if} \quad n > N(\nu). \tag{8}$$

L. Leindler in [4], [5] (see also [6], p. 154) proved the following relation generalizing results (2) and (3).

$$\left.\begin{aligned} &(9) \qquad H(\beta, p, \omega_\alpha) \equiv H^{\omega_\alpha} && \text{for} \quad 0 < \alpha < 1; \\ &(10) \qquad H(\beta, p, \omega_1) \equiv (H^{\omega_1})^* && \text{for} \quad \alpha = 1 \end{aligned}\right\} \text{ if } \quad \beta > \alpha p.$$

It is clear that in order to get coefficient conditions of type (6) for $f \in H(\beta, p, \omega_\alpha)$ instead of $H(\beta, p, \delta^\alpha)$ it is sufficient to generalize the mentioned Boas results to class H^{ω_α}. These results are formulated in the next paragraph.

3. Theorems

Throughout the rest of this paper $g(x)$, $f(x)$, $\varphi(x)$ will denote continuous 2π periodic functions; furthermore $g(x)$ and $f(x)$ always denote the sum of sine series and cosine series, respectively. And $\varphi(x)$ denotes the sum of either sine or cosine series while λ_n will denote the Fourier coefficients of $g(x)$, $f(x)$ or $\varphi(x)$.

Theorem 1. *Let* $\lambda_n \geqq 0$. *Then* $\varphi \in H^{\omega_\alpha}$ $(0<\alpha<1)$ *if and only if*

$$\sum_{k=n}^{\infty} \lambda_k = O(\omega_\alpha(1/n)), \tag{11}$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n\omega_\alpha(1/n)). \tag{12}$$

Theorem 2. *Let* $\lambda_n \geqq 0$. *Then*

$$g \in H^{\omega_1} \tag{13a}$$

if and only if

$$\sum_{k=1}^{n} k\lambda_k = O(n\omega_1(1/n)). \tag{13b}$$

Theorem 3. *Let* $\lambda_n \geqq 0$. *Then*

$$\varphi \in (H^{\omega_1})^* \tag{14a}$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\omega_1(1/n)). \tag{14b}$$

Theorem 4. *Let* $\lambda_n \geqq 0$. *Then*

(15a) $$f \in H^{\omega_1}$$

if and only if

(15b) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_1(1/n)) \quad \textit{and} \quad \sum_{k=1}^{n} k\lambda_k \sin kx = O(n\omega_1(1/n)).$$

Theorem 5. *Let* $\lambda_n \geqq 0$. *Then*

(16a) $$f \in H^{\omega_0}$$

if and only if

(16b) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_0(1/n)).$$

Furthermore

(17) $$g \in H^{\omega_0}$$

implies

(18) $$\sum_{k=1}^{n} k\lambda_k = O(n\omega_0(1/n)),$$

and from

(19) $$\sum_{k=n}^{\infty} \lambda_k = O(\omega_0(1/n))$$

(20) $$g \in H^{\omega_0}$$

follows.

Remark 1. Combining relation (9) of L. Leindler and Theorem 1 we get generalizations of the relations under (6). Namely that if $\lambda_n \geqq 0$ then for $0<\alpha<1$ with $\beta>\alpha p$

$$\varphi \in H(\beta, p, \omega_\alpha)$$

if and only if

$$\sum_{k=n}^{\infty} \lambda_k = O(\omega_\alpha(1/n)),$$

or equivalently

$$\sum_{k=1}^{n} k\lambda_k = O(n\omega_\alpha(1/n)).$$

Remark 2. Using Theorem 1 we can prove the following result.
If $h \in H^{\omega_\alpha}$ $(0<\alpha<1)$ and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

is the Fourier series of $h(x)$ and $a_k \geqq 0$, $b_k \geqq 0$ then

$$\|s_n(x)-h(x)\| = O(\omega_\alpha(1/n)). \tag{21}$$

It should be noted that for arbitrary a_k, b_k according to the well-known Lebesgue result only

$$\|s_n(x)-h(x)\| = O(\omega_\alpha(1/n))\cdot \log n$$

can be obtained.

The proof of this remark is very simple. Really, consider

$$\|s_n(x)-h(x)\| \leqq \|s_n(x)-\sigma_n(x)\| + \|\sigma_n(x)-h(x)\| = I + II.$$

Using (12) and the fact that

$$s_n(x)-\sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n}(ka_k \cos kx + kb_k \sin kx)$$

I can be estimated as follows

$$I \leqq \frac{1}{n+1}\sum_{k=1}^{n} k(a_k+b_k) = O(\omega_\alpha(1/n)).$$

Here we used that if $h \in H^{\omega_\alpha}$, then both its sine-part and cosine-part also belong to H^{ω_α}. And finally from (9)

$$II = O(\omega_\alpha(1/n)).$$

Thus we have (21).

4. Lemmas

Lemma 1. (Lemma 2.6 of [6], p. 39.) *For any nonnegative sequence* $\{a_n\}$ *the inequality*

$$\sum_{n=1}^{m} a_n \leqq K a_m \quad (m = 1, 2, \ldots;\ K > 0) \tag{22}$$

holds if and only if there exist a positive number c and a natural number μ *such that for any n*

$$a_{n+1} \geqq c a_n \tag{23}$$

and

$$a_{n+\mu} \geqq 2a_n \tag{24}$$

are valid.

Lemma 2. *If* $\mu_k \geqq 0$ *and* $\delta > \beta > 0$ *then*

(25a)
$$\sum_{k=1}^{n} k^\delta \mu_k = O\big(n^\delta \omega_{\delta-\beta}(1/n)\big)$$

is equivalent to

(25b)
$$\sum_{k=n}^{\infty} \mu_k = O\big(\omega_{\delta-\beta}(1/n)\big).$$

Proof. First we suppose that (25b) is true. By using Abel-rearrangement ([1], p. 71) we have

(26)
$$\sum_{k=1}^{n} \mu_k k^\delta \leqq \sum_{k=1}^{n} \big(k^\delta-(k-1)^\delta\big) \sum_{\nu=k}^{\infty} \mu_\nu + 2 \sum_{\nu=1}^{\infty} \mu_\nu = I + II.$$

It is obvious by (25b) and the definition of $\omega_{\delta-\beta}(t)$ that II does not exceed $K \cdot n^\delta \omega_{\delta-\beta}(1/n)$*).

And I can be estimated as follows

(27)
$$I \leqq K_1 \sum_{k=1}^{n} k^{\delta-1} \omega_{\delta-\beta}(1/k) \leqq K_2 \sum_{m=1}^{[\log n]} (2^m)^{\delta-1} \cdot 2^m \omega_{\delta-\beta}(1/2^m) =$$
$$= K_2 \sum_{m=1}^{[\log n]} 2^{m\delta} \omega_{\delta-\beta}(1/2^m) \leqq K_3 n^\delta \omega_{\delta-\beta}(1/n),$$

where the last estimation can be obtained by using property (7) of $\omega_\alpha(\delta)$ and Lemma 1. So taking into account (26) and (27), (25a) really follows from (25b). Now we suppose (25a).

Again Abel-rearrangement gives that

(28)
$$\sum_{k=m}^{n} \mu_k = \sum_{k=m}^{n} k^\delta \mu_k k^{-\delta} = \sum_{k=m}^{n} \big(k^{-\delta}-(k+1)^{-\delta}\big) \sum_{l=1}^{k} l^\delta \mu_l + n^{-\delta} \sum_{k=1}^{n} k^\delta \mu_k.$$

Making n tend to infinity from (25a) and (28) we get

(29)
$$\sum_{k=m}^{\infty} \mu_k = \sum_{k=m}^{\infty} \big(k^{-\delta}-(k+1)^{-\delta}\big) \sum_{l=1}^{k} l^\delta \mu_l = I_1.$$

I_1 can be estimated by (25a) as follows

(30)
$$I_1 \leqq K_1 \sum_{k=m}^{\infty} k^{-1} \omega_{\delta-\beta}(1/k) \leqq K_2 \sum_{n=[\log m]}^{\infty} \omega_{\delta-\beta}(1/2^n) \leqq K_3 \omega_{\delta-\beta}(1/m).$$

The last step can be derived from property (8) of $\omega_\alpha(\delta)$. From (29) and (30) we have (25b). Thus Lemma 2 is completely proved.

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*) $K, K_1, K_2, \ldots$ will denote positive absolute constants.

Lemma 3. *If* $\mu_k \geqq 0$, $\sum_{k=1}^{\infty} \mu_k < \infty$ *and* $0 < \beta \leqq 1$, *then*

(31a) $$\sum_{k=1}^{\infty} \mu_k (1-\cos kx) = O(\omega_\beta(x))$$

is equivalent to

(31b) $$\sum_{k=n}^{\infty} \mu_k = O(\omega_\beta(1/n)).$$

Proof. Supposing first (31a), we have that

(32) $$\sum_{k=1}^{[1/x]} k^2 \mu_k \frac{1-\cos kx}{k^2 x^2} = O(x^{-2}\omega_\beta(x))$$

holds for any positive x.

Since $K \geqq t^{-2}(1-\cos t)\downarrow$ on $(0, 1)$, from (32) it follows that

(33) $$\sum_{k=1}^{[1/x]} k^2 \mu_k = O(x^{-2}\omega_\beta(x)).$$

Putting $x = 1/n$ we get

(34) $$\sum_{k=1}^{n} k^2 \mu_k = O(n^2 \omega_\beta(1/n))$$

which, by using Lemma 2, is equivalent to

(35) $$\sum_{k=n}^{\infty} \mu_k = O(\omega_\beta(1/n))$$

which proves (31a)$\Rightarrow$(31b).

Now we suppose that (31b) is valid. But (31b), by using Lemma 2, is equivalent to

(36) $$\sum_{k=1}^{n} k^2 \mu_k = O(n^2 \omega_\beta(1/n)).$$

Using (31b) and (36) we have that

$$\sum_{k=1}^{\infty} \mu_k (1-\cos kx) \leqq \sum_{k=1}^{[1/x]} \mu_k (1-\cos kx) + 2 \sum_{k=[1/x]}^{\infty} \mu_k =$$

$$= x^2 \sum_{k=1}^{[1/x]} k^2 \mu_k \frac{1-\cos kx}{k^2 x^2} + 2 \sum_{k=[1/x]}^{\infty} \mu_k = O(\omega_\beta(x))$$

which gives that (31b) really implies (31a). Thus Lemma 3 is completed.

5. Proofs of the theorems

Proof of Theorem 1. We detail the proof just for cosine, the line of the proof for sine is very similar.

Suppose that $f\in H^{\omega_\alpha}$ $(0<\alpha<1)$. Then since by the Paley's theorem from the continuity of $f(x)$ it follows that $\sum_{k=1}^{\infty}\lambda_k<\infty$, we have

$$|f(x)-f(0)| = \sum_{k=1}^{\infty}\lambda_k(1-\cos kx) = O(\omega_\alpha(x)). \tag{37}$$

By Lemma 3 the right-hand side equality of (37) is equivalent to (11). In virtue of Lemma 2 (11) is equivalent to (12). Thus the necessary part of Theorem 1 is proved. Now we suppose that (11) holds and put

$$|f(x+2h)-f(x)| = \Big|\sum_{k=1}^{\infty}\lambda_k[\cos k(x+2h)-\cos kx]\Big| = \tag{38}$$

$$= 2\Big|\sum_{k=1}^{\infty}\lambda_k\sin k(x+h)\cdot\sin kh\Big| \leqq 2\sum_{k=1}^{\infty}\lambda_k|\sin kh| \leqq 2\sum_{k=1}^{[1/h]}\lambda_k\sin kh+2\sum_{k=[1/h]}^{\infty}\lambda_k.$$

The second term in the last row of (38) is $O(\omega_\alpha(h))$ (see (11)). The first one can be handled as follows:

$$\sum_{k=1}^{[1/h]}\lambda_k\cdot\sin kh = h\sum_{k=1}^{[1/h]}k\lambda_k\frac{\sin kh}{kh} \leqq K\cdot h\Big(\sum_{k=1}^{[1/h]}k\lambda_k\Big) = O(\omega_\alpha(h)).$$

In the last step we used again (12) and Lemma 2. So from (38) we have $f\in H^{\omega_\alpha}$. The proof of Theorem 1 for cosine series is completed.

Proof of Theorem 2. Let $g\in H^{\omega_1}$. Using

$$|g(x)| \leqq K\omega_1(x),$$

term by term integration gives

$$\Big|\int_0^x g(t)\,dt\Big| = \sum_{k=1}^{\infty}k^{-1}\lambda_k(1-\cos kx) = O(x\omega_1(x)). \tag{39}$$

From (39) it follows that

$$\sum_{k=1}^{[1/x]}k^{-1}\lambda_k(1-\cos kx) = O(x\omega_1(x)). \tag{40}$$

(40) can be written as follows

$$x^2\sum_{k=1}^{[1/x]}k\lambda_k\frac{1-\cos kx}{k^2x^2} = O(x\omega_1(x)). \tag{41}$$

By using the same argument as before from (41) we get

$$\sum_{k=1}^{[1/x]} k\lambda_k = O\left(\frac{1}{x}\,\omega_1(x)\right). \tag{42}$$

Putting $x=1/n$ we have from (42)

$$\sum_{k=1}^{n} k\lambda_k = O\big(n\omega_1(1/n)\big)$$

which proves that from (13a) follows (13b). Now we suppose that (13b) is fulfilled. Put

$$|g(x+2h)-g(x)| = \Big|\sum_{k=1}^{\infty} \lambda_k\,[\sin k(x+2h)-\sin kx]\Big| = \tag{43}$$

$$= 2\Big|\sum_{k=1}^{\infty} \lambda_k \cos k(x+h)\sin kh\Big| \leqq 2\sum_{k=1}^{[1/h]} \lambda_k \sin kh + \sum_{k=[1/h]}^{\infty} \lambda_k = I+II.$$

By using (13b) we have that

$$I = 2h\sum_{k=1}^{[1/h]} k\lambda_k \frac{\sin kh}{kh} = h\cdot O\left(\frac{1}{h}\,\omega_1(h)\right) = O\big(\omega_1(h)\big). \tag{44}$$

For II to be estimated by $K\omega_1(h)$ we can use the same argument as in the second part of the proof Lemma 2. Namely taking $\delta=1$ and $\delta-\beta=1$ we get that (13b) implies

$$\sum_{k=n}^{\infty} \lambda_k = O\big(\omega_1(1/n)\big)$$

which gives that

$$II = O\big(\omega_1(h)\big). \tag{45}$$

Thus (43), (44) and (45) give that $g\in H^{\omega_1}$. Theorem 2 is completed.

Proof of Theorem 3.

First we prove the theorem for cosine series. Suppose that (14a) holds, that is,

$$|f(x+h)+f(x-h)-2f(x)| \leqq K\omega_1(h)$$

from which we get

$$|f(h)-f(0)| \leqq K\omega_1(h) \tag{46}$$

in other words

$$\sum_{k=1}^{\infty} \lambda_k(1-\cos kh) = O\big(\omega_1(h)\big) \tag{47}$$

holds.

From (47) by using Lemma 3, (14b) follows, that was to be proved. Now we assume (14b) and estimate the following difference by using Lemma 3 at the last step:

$$|f(x+2h)+f(x-2h)-2f(x)| = 4\left|\sum_{k=1}^{\infty} \lambda_k \sin^2 kh \cos kx\right| \leqq \tag{48}$$

$$\leqq 4\sum_{k=1}^{\infty} \lambda_k \sin^2 kh = 2\sum_{k=1}^{\infty} \lambda_k(1-\cos 2kh) = O(\omega_1(h)).$$

Thus the proof of Theorem 3 is completed for cosine series. The proof for sine series in direction from (14b) to (14a) can be done in the same way as for cosine series, since

$$|g(x+2h)+g(x-2h)-2g(x)| = 4\left|\sum_{k=1}^{\infty} \lambda_k \sin kx \sin^2 kh\right|. \tag{49}$$

So we detail only the other direction. Suppose that

$$g\in(H^{\omega_1})^*, \tag{50}$$

that is,

$$|g(x+h)+g(x-h)-2g(x)| = O(\omega_1(h)). \tag{51}$$

Write (51) in the following form

$$2\left|\sum_{k=1}^{\infty} \lambda_k \sin kx(1-\cos kh)\right| = O(\omega_1(h)). \tag{52}$$

By integrating term by term in (52) we get

$$\sum_{k=1}^{\infty} \lambda_k \frac{1-\cos kx}{k}(1-\cos kh) = O(x\omega_1(h)). \tag{53}$$

From (53) we have

$$\sum_{k=1}^{\infty} x^2 k\lambda_k \frac{1-\cos kx}{k^2x^2}(1-\cos kh) = O(x\omega_1(h)). \tag{54}$$

Using (54) it follows that

$$\sum_{k=1}^{[1/x]} xk\lambda_k(1-\cos kh) = O(\omega_1(h)). \tag{55}$$

Putting $h=x$ in (55)

$$\sum_{k=1}^{[1/h]} hk\lambda_k(1-\cos kh) = O(\omega_1(h)) \tag{56}$$

can be obtained which gives

$$\sum_{k=1}^{[1/h]} h^3 k^3 \lambda_k \frac{1-\cos kh}{k^2 h^2} = O(\omega_1(h)). \tag{57}$$

From (57) taking $h=1/n$

$$\sum_{k=1}^{n} k^3 \lambda_k = O(n^3 \omega_1(1/n)) \tag{58}$$

follows.

By using Lemma 2 (58) implies (14b), which was to be proved. Thus Theorem 3 is completely proved.

Proof of Theorem 4. First we prove the necessity of the conditions, namely we suppose that (15a) holds.
From Theorem 3 using the relation $H^{\omega_1} \subset (H^{\omega_1})^*$ it follows that

$$\sum_{k=n}^{\infty} \lambda_k = O(\omega_1(1/n)). \tag{59}$$

So it remains just to prove

$$\Big\|\sum_{k=1}^{n} k\lambda_k \sin kx\Big\| = O(n\omega_1(1/n)). \tag{60}$$

Set

$$|f(x+h)-f(x)| = 2\sum_{k=1}^{[1/h]} \lambda_k \sin k(x+h) \sin kh + O\Big(\sum_{k=[1/h]}^{\infty} \lambda_k\Big).$$

From (15a) and (59) we get

$$\Big\|\sum_{k=1}^{[1/h]} \lambda_k \sin k(x+h) \sin kh\Big\| = O(\omega_1(h)). \tag{61}$$

Since $\sin kh = kh + O(k^3h^3)$ we have from (61) that

$$\Big\|h\sum_{k=1}^{[1/h]} \lambda_k k \sin k(x+h) + h^3 \sum_{k=1}^{[1/h]} \lambda_k k^3 \sin k(x+h)\Big\| = O(\omega_1(h)) \tag{62}$$

and having in view Lemma 2 we get

$$\Big\|h^3 \sum_{k=1}^{[1/h]} \lambda_k k^3 \sin k(x+h)\Big\| = O(\omega_1(h)). \tag{63}$$

Using (63) we have from (62)

$$\Big\|h \sum_{k=1}^{[1/h]} k\lambda_k \sin k(x+h)\Big\| = O(\omega_1(h)). \tag{64}$$

Since $\sin k(x+h)=\sin kx-O(k^2h^2)\sin kx+O(kh)\cos kx$, (64) has the following form

$$\left\|h\sum_{k=1}^{[1/h]}k\lambda_k\sin kx-h\sum_{k=1}^{[1/h]}k\lambda_k O(k^2h^2)\sin kx+\right. \tag{65}$$

$$\left.+h\sum_{k=1}^{[1/h]}k\lambda_k O(kh)\cos kx\right\|=O(\omega_1(h)).$$

Taking into account that from (59) by using Lemma 2

$$\sum_{k=1}^{n}k^3\lambda_k=O\big(n^3\omega_1(1/n)\big) \tag{66}$$

follows, the norm of the second term in the left-hand side of (65) can be estimated as follows

$$\left\|h\sum_{k=1}^{[1/h]}k\lambda_k O(k^2h^2)\sin kx\right\|\leqq Kh^3\sum_{k=1}^{[1/h]}k^3\lambda_k=O\big(\omega_1(h)\big). \tag{67}$$

Similarly by using

$$\sum_{k=1}^{n}k^2\lambda_k=O\big(n^2\omega_1(1/n)\big)$$

instead of (66) we can get that the magnitude of the third term of (65) in norm is $O(\omega_1(h))$. Using this last estimation and (65), (67) we have (60).

The sufficiency of conditions (15b) can be proved in very similar way as the necessity, so we omit it. Thus Theorem 4 is completed.

Proof of Theorem 5. Let $f(x)=\sum_{k=1}^{\infty}\lambda_k\cos kx$ and suppose that $f\in H^{\omega_0}$. Then we have

$$|f(h)-f(0)|\leqq K\omega_0(h),$$

that is,

$$\sum_{k=1}^{\infty}\lambda_k(1-\cos kh)\leqq K\omega_0(h).$$

Integrating both sides on $(0,x)$ we have

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{k}(kx-\sin kx)\leqq Kx\omega_0(x). \tag{68}$$

Since $kx-\sin kx\geqq 0$ so we have from (68)

$$\sum_{k=2n}^{\infty}\frac{\lambda_k}{k}(kx-\sin kx)\leqq Kx\omega_0(x). \tag{69}$$

Putting $1/n$ for x and taking into account that

$$\frac{k}{n}-\sin\frac{k}{n} \geqq \frac{1}{2}\frac{k}{n} \quad \text{for} \quad k \geqq 2n$$

we get

(70) $$\sum_{k=2n}^{\infty} \lambda_k \leqq K_1\omega_0(1/n)$$

which gives (16b).

Now we suppose that (16b) holds and we prove $f \in H^{\omega_0}$. First we note that we can notice that the first part of the proof of Lemma 2 remains valid if we take $\delta-\beta=0$ and $\delta=1$. So from (16b) we have

(71) $$\sum_{k=1}^{n} k\lambda_k = O\big(n\omega_0(1/n)\big).$$

And now estimate the following difference using (16b) and (71).

$$|f(x+2h)-f(x)| = \Big|\sum_{k=1}^{\infty} \lambda_k[\cos k(x+2h)-\cos kx]\Big| =$$

$$= 2\Big|\sum_{k=1}^{\infty} \lambda_k \sin k(x+h) \sin kh\Big| \leqq 2\Big|\sum_{k=1}^{\infty} \lambda_k \sin kh\Big| =$$

$$\leqq 2\sum_{k=1}^{[1/h]} \lambda_k \sin kh + \sum_{k=[1/h]}^{\infty} \lambda_k = O\big(\omega_0(h)\big)$$

which proves that (16b) implies (16a).

Now we prove (18) from (17).

Suppose that $g \in H^{\omega_0}$. Using the estimation

(72) $$|g(x)| \leqq K\omega_0(x),$$

term by term integration gives from (72) that

(73) $$\sum_{k=1}^{\infty} \frac{\lambda_k}{k}(1-\cos kx) \leqq Kx\omega_0(x).$$

From (73)

$$\sum_{k=1}^{[1/x]} k\lambda_k \leqq K\frac{\omega_0(x)}{x}$$

follows as at the proof of (33) which taking $x=1/n$ gives (18). The proof of (20) from (19) can be done in the very same way as (16a) from (16b), so we omit it. Theorem 5 is completed.

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Starke Approximation von Orthogonalreihen mit Cesàroverfahren

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Es sei $\{\varphi_n(x)\}$ ein auf dem Intervall (a, b) definiertes Orthonormalsystem. Wir untersuchen die Orthonormalreihe

$$(1)\qquad \sum_{n=0}^{\infty} c_n \varphi_n(x) \quad c_n \in \mathbf{R} \quad \text{mit} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

Mit dem wohlbekannten Satz von Riesz—Fischer folgt, daß Reihe (1) in L^2 gegen eine quadratisch integrierbare Funktion f konvergiert. Wir nennen die Partialsummen von Reihe (1) $s_n(x)$ und die (C, α)-Mittel $\sigma_n^\alpha(x)$.

Eine Verallgemeinerung der Cesàro-Verfahren führt zu den Hausdorffverfahren. Ein Hausdorffverfahren ist ein lineares Limitierungsverfahren, das mit Hilfe einer beliebigen Diagonalmatrix $\mu = \mu_\nu$ und der Differenzenmatrix Δ wie folgt definiert ist.

$$H(\Delta, \mu_\nu) = \Delta \cdot \mu \cdot \Delta \quad \text{mit} \quad \Delta = \left((-1)^\nu \binom{n}{\nu}\right).$$

Die Matrixelemente $h_{n\nu}$ der Hausdorffmatrix $H(\Delta, \mu_\nu)$ haben folgende Darstellung

$$h_{n\nu} = \begin{cases} \binom{n}{\nu} \sum_{k=0}^{n-\nu} (-1)^k \binom{n-\nu}{k} \mu_{\nu+k} & \text{für} \quad 0 \leqq \nu \leqq n \quad (n = 0, 1, \ldots) \\ 0 & \text{sonst.} \end{cases}$$

Über die Regularität von Hausdorffverfahren gilt folgender

Satz. *Das Hausdorffverfahren H ist genau dann regulär, wenn $\{\mu_n\}$ eine reguläre Momentenfolge ist, d. h.* $\mu_n = \int_0^1 t^n \, d\mu(t)$ *mit* $\mu(t) \in \mathrm{BV}\,[0, 1]$ *und* $\mu(+0) = \mu(0) = 0$, $\mu(1) = 1$.

L. Leindler zeigte in [2]

Eingegangen am 8 Februar 1988.

Satz I. *Es sei* $0<\delta<1$ *und*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\delta} < \infty, \tag{2}$$

dann gilt $f(x)-\sigma_n^1(x)=o_x(n^{-\delta})$ *f. ü. auf* (a, b).

G. Sunouchi [5] verallgemeinerte Satz I zur starken Approximation wie folgt:

Satz II. *Es sei* $0<\delta<1$ *und es gelte* (2). *Dann gilt*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|f(x)-s_\nu(x)|^k\right\}^{1/k} = o_x(n^{-\delta})$$

f. ü. auf (a, b) *für* $\alpha>0$ *und* $0<k<\delta^{-1}$ *wobei* $A_n^\alpha=\binom{n+\alpha}{n}$.

Leindler [3] wiederum verallgemeinerte das Ergebnis von Sunouchi wie folgt:

Satz III. *Es sei* $0<\delta<1, \alpha>0, 0<k<\delta^{-1}$ *und* $\beta>-\min(1/2, 1/k, \alpha/k)$. *Dann folgt aus* (2)

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^k\right\}^{1/k} = o_x(n^{-\delta}) \quad \textit{f.ü. auf} \quad (a, b).$$

Für schwache äußere Verfahren (d. h. α sehr klein) kann sich β offensichtlich nur in einem sehr engen negativen Bereich bewegen. Bezogen auf diese Problematik zeigte Leindler [4]

Satz IV. *Es sei* $0<\alpha<1$, $\beta>-1/2$, $0<\delta<\alpha/2$ *und*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\delta+1-\alpha} < \infty. \tag{3}$$

dann gilt

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^2\right\}^{1/2} = o_x(n^{-\delta}) \quad \textit{f.ü. auf} \quad (a, b).$$

Der Satz von Leindler gibt Kriterien für die starke Approximation an, speziell für schwache äußere (C, α)-Verfahren mit dem Exponent 2. Es stellt sich die Frage: Welche Aussagen kann man für größeren Exponenten $(k>2)$ treffen? Eine Antwort liefert der folgende Satz.

Satz 1. *Es sei* $0<\alpha<1$, $k>2$, $0<\delta<\alpha/2$, $\beta>-1/2+\alpha(1/2-1/k)$, *und es gelte* (3). *Dann gilt*

$$\left(\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|\sigma_\nu^\beta(x)-f(x)|^k\right)^{1/k} = o_x(n^{-\delta})$$

f. ü. auf (a, b).

Zur Vorbereitung des Beweises von Satz 1 beweisen wir einen allgemeinen limitierungstheoretischen Satz über Hausdorffverfahren, der besagt: Schaltet man bei der starken Approximation einer beliebigen Reihe vor das innere Verfahren ein reguläres Hausdorffverfahren, so kann man auch auf größere k-Parameterwerte (Exponent) schließen. Der Schluß auf kleinere k-Parameterwerte kann immer in einfacher Weise mit der Hölderschen Ungleichung vollzogen werden.

Satz 2. *Es sei* $k_1 > k_2 \geqq 1$; $p > 1$,

$$g_n = \int_0^1 t^n g(t)\,dt \quad \textit{mit} \quad g(t) \in L^p[0, 1] \quad \textit{und} \quad g_0 = 1$$

$$h_n = \int_0^1 t^n h(t)\,dt \quad \textit{mit} \quad h(t) \in L[0, 1], h(t) > 0 \quad \textit{für} \quad t \in [0, 1].$$

Zusätzliche gelte für die Funktionen $g(t)$ *und* $h(t)$

$$z(t) := \begin{cases} |g(t)|^{k_1(1-p+(p/k_2))}\, h(t)^{(k_2-k_1)/k_2} & \textit{für} \quad g(t) \neq 0 \\ 0 & \textit{sonst}, \end{cases}$$

$z(t) \in L[0, 1]$ *und* $z(t) = o(t^{-r})$ *für* $t \in [0, \varepsilon]$ *mit* $\varepsilon > 0$ *und* $0 < r < 1$. *Q sei eine beliebige Limitierungsmatrix, G bzw. H seien Hausdorffmatrizen, die durch die Momentenfolgen* $\{g_n\}$ *bzw.* $\{h_n\}$ *gegeben sind. Wir bezeichnen die Matrixelemente von H mit* $h_{n\nu}$. *Dann folgt für eine beliebige Folge* $\{s_n\}$ *und für alle* $\delta > 0$ *mit* $k_1\delta < 1 - r$ *aus*

$$\Big(\sum_{\nu=0}^{n} h_{n\nu}\, |Q(s_\nu) - s|^{k_2}\Big)^{1/k_2} = o(n^{-\delta}),$$

daß

$$\Big(\sum_{\nu=0}^{n} h_{n\nu}\, |GQ(s_\nu) - s|^{k_1}\Big)^{1/k_1} = o(n^{-\delta}).$$

Zum Beweis von Satz 2 benötigen wir das folgende Lemma.

Lemma 1. *Es sei* $k_1 > k_2 \geqq 1$; $p > 1$. *Die Hausdorffverfahren G und H seien definiert wie in Satz 2*

$$z_n := \int_0^1 t^n z(t)\,dt \quad \textit{mit} \quad z(t) \quad \textit{aus Satz 2.}$$

Das Hausdorffverfahren Z sei definiert durch die Momentenfolge $\{z_n\}$. *Dann gilt für jede beliebige Folge* $\{s_\nu\}$:

$$\big|(G(s_\nu))_n\big|^{k_1} \leqq K\big(H(|s_\nu|^{k_2})\big)_n^{(k_1/k_2)-1} \cdot \big(Z(|s_\nu|^{k_2})\big)_n \quad (n = 0, 1, 2, \ldots),$$

wobei K eine Konstante ist.

Lemma 2. *Es sei $\{s_n\}$ eine Folge mit $s_n \to 0$ für $n \to \infty$. Das Limitierungsverfahren A sei gegeben durch die Matrix $(a_{n\nu})$. Dann gilt: Das Verfahren A limitiert die Folge $\{s_n\}$ zum Wert Null, d. h. $A(s_n) \to 0$, falls folgende Bedingungen erfüllt sind:*

(I) $\sum_{\nu=0}^{\infty} |a_{n\nu}| < K$, *wobei K unabhängig von n ist,* $n=0, 1, 2, \ldots$

(II) $a_{n\nu} \to 0$ *für jedes feste ν und für* $n \to \infty$.

Beweis. Siehe ([1] Seite 43—46).

Hilfssatz 1.

$$\binom{n}{\nu} \int_0^1 t^\nu (1-t)^{n-\nu} t^{-r} \, dt = O(1)(\nu+1)^{-r}/(n+1)^{1-r} \quad \text{für} \quad 0 < \nu \leqq n,$$

$$n = 0, 1, \ldots \quad \text{und} \quad 0 < r < 1.$$

Beweis. Ergibt sich leicht unter Berücksichtigung der Integraldarstellung der Matrixelemente der Cesàro-Verfahren.

Beweis von Lemma 1. Wir setzen

$$f_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu} s_\nu \quad (n = 0, 1, 2, \ldots).$$

Zunächst führen wir eine Abschätzung durch, die wir im letzten Beweisschritt benutzen werden.

$$|f_n(t)|^{k_2} \leqq \sum_{\nu=0}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu} |s_\nu|^{k_2} \left(\sum_{\nu=0}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu} \right)^{k_2-1} =$$

$$= \sum_{\nu=0}^{n} \binom{n}{\nu} t^\nu (1-t)^{n-\nu} |s_\nu|^{k_2}.$$

Also erhalten wir

$$(*) \qquad \int_0^1 h(t) \, |f_n(t)|^{k_2} \, dt \leqq \sum_{\nu=0}^{n} \int_0^1 \binom{n}{\nu} t^\nu (1-t)^{n-\nu} h(t) \, dt \, |s_\nu|^{k_2} = (H(|s_\nu|^{k_2}))_n.$$

In analoger Weise ergibt sich

$$(**) \qquad \int_0^1 z(t) \, |f_n(t)|^{k_2} \, dt \leqq (Z(|s_\nu|^{k_2}))_n.$$

Es sei M die Menge aller Punkte t, für die gilt $t\in[0, 1]$ und $g(t)\neq 0$.

$$|G(s_\nu)_n|^{k_2} \leqq \Big\{\int_0^1 |g(t)|^p\,dt\Big\}^{k_2-1}\Big\{\int_M |g(t)|^{(1-p(1-1/k_2))k_2}\,|f_n(t)|^{k_2}\,dt\Big\} =$$

$$= K^{k_2-1}\int_M |g(t)|^{(1-p(1-1/k_2))k_2}\,|f_n(t)|^{k_2}\,dt.$$

$$|G(s_\nu)_n|^{k_2} \leqq K^{k_2-1}\int_M h(t)^{(k_2-k_1)/k_1}\,|f_n(t)|^{k_2(k_2/k_1)}\,|g(t)|^{k_2(1-p+(p/k_2))}\times$$

$$\times h(t)^{(k_1-k_2)/k_1}\,|f_n(t)|^{k_2(k_1-k_2)/k_1}\,dt \leqq$$

$$\leqq K^{k_2-1}\Big\{\int_M h(t)^{(k_2-k_1)/k_2}\,|f_n(t)|^{k_2}\,|g(t)|^{k_1(1-p+(p/k_2))}\,dt\Big\}^{k_2/k_1}\times$$

$$\times\Big\{\int_M h(t)\,|f_n(t)|^{k_2}\,dt\Big\}^{(k_1-k_2)/k_1}.$$

Mit $z(t)=|g(t)|^{k_1(1-p+(p/k_2))}h(t)^{(k_2-k_1)/k_2}$ und mit den Abschätzungen $(*)$ und $(**)$ folgt

$$|G(s_\nu)_n|^{k_2} \leqq K^{k_2-1}\big(H(|s_\nu|^{k_2})\big)_n^{(k_1-k_2)/k_1}\big(Z(|s_\nu|^{k_2})\big)_n^{k_2/k_1}.$$

Indem man diese Ungleichung mit k_1/k_2 potenziert erhält man die Behauptung.

Beweis von Satz 2. Zum Beweis von Satz 2 reicht es aus, den Fall $Q=1$ zu betrachten. Es sei $\{s_n\}$ eine beliebige Folge, und es gelte

$$\Big\{\sum_{\nu=0}^{n} h_{n\nu}\,|s_\nu-s|^{k_2}\Big\}^{1/k_2} = o(n^{-\delta}).$$

Mit $z_{n\nu}$ bezeichnen wir die Matrixelemente des Limitierungsverfahrens Z, das durch die Momentenfolge $\{z_n\}$ aus Lemma 1 gegeben ist.

Da das Hausdorffverfahren G regulär ist, gilt insbesondere

$$\big(G(s_j)\big)_\nu - s = \big(G(s_j-s)\big)_\nu.$$

Unter Verwendung von Lemma 1 erhalten wir

$$n^{k_1\delta}\sum_{\nu=0}^{n} h_{n\nu}\big|\big(G(s_j)\big)_\nu - s\big|^{k_1} = n^{k_1\delta}\sum_{\nu=0}^{n} h_{n\nu}\big|\big(G(s_j-s)\big)_\nu\big|^{k_1} =$$

$$= O(1)\,n^{k_1\delta}\sum_{\nu=0}^{n} h_{n\nu}\{H(|s_j-s|^{k_2})\}_\nu^{(k_1/k_2)-1}\{Z(|s_j-s|^{k_2})\}_\nu =$$

$$= O(1)\,n^{k_1\delta}\sum_{\nu=0}^{n} h_{n\nu}\{Z(|s_j-s|^{k_2})\}_\nu\cdot(\nu+1)^{\delta(k_2-k_1)}.$$

Denn nach Voraussetzung gilt $\big(H(|s_j-s|^{k_2})\big)_\nu = o(\nu^{-k_2\delta})$.

Die Hausdorffverfahren H und Z sind vertauschbar. Somit gilt

$$n^{k_1\delta}\sum_{\nu=0}^{n} h_{n\nu}(Z(|s_j-s|^{k_2}))_\nu(\nu+1)^{-k_1\delta+k_2\delta}=$$

$$=O(1)n^{k_1\delta}\sum_{\nu=0}^{n} z_{n\nu}\sum_{j=0}^{\nu} h_{\nu j}|s_j-s|^{k_2}(\nu+1)^{-k_1\delta+k_2\delta}=$$

$$=O(1)\sum_{\nu=0}^{n} n^{k_1\delta}(\nu+1)^{-k_1\delta}z_{n\nu}t_\nu$$

mit

$$t_\nu=(\nu+1)^{k_2\delta}\sum_{j=0}^{\nu} h_{\nu j}|s_j-s|^{k_2}.$$

Nach Voraussetzung gilt $t_\nu=o(1)$. Somit haben wir unter Beachtung von Lemma 2 zu zeigen:

(I) $$\sum_{\nu=0}^{\infty}|n^{k_1\delta}(\nu+1)^{-k_1\delta}z_{n\nu}|<K \quad \text{für} \quad n=0,1,2,\ldots$$

(II) $$\{n^{k_1\delta}(\nu+1)^{-k_1\delta}z_{n\nu}\}\to 0 \quad (n\to\infty) \quad \text{für} \quad \nu=0,1,2,\ldots.$$

Zu (I). Mit der Integraldarstellung der Hausdorffschen Matrixelemente $z_{n\nu}$ ergibt sich somit:

$$\sum_{\nu=0}^{\infty} n^{k_1\delta}(\nu+1)^{-k_1\delta}z_{n\nu}=\sum_{\nu=0}^{n} n^{k_1\delta}(\nu+1)^{-k_1\delta}\binom{n}{\nu}\int_0^1 t^\nu(1-t)^{n-\nu}z(t)\,dt=$$

$$=n^{k_1\delta}\sum_{\nu=0}^{n}\binom{n}{\nu}\int_0^\varepsilon t^\nu(1-t)^{n-\nu}z(t)\,dt(\nu+1)^{-k_1\delta}+$$

$$+n^{k_1\delta}\sum_{\nu=0}^{n}\binom{n}{\nu}\int_\varepsilon^1 t^\nu(1-t)^{n-\nu}z(t)\,dt(\nu+1)^{-k_1\delta}=\Sigma_1+\Sigma_2.$$

Nach Voraussetzung gilt $z(t)=o(t^{-r})$ für $t\in[0,\varepsilon]$. Somit erhalten wir mit Hilfssatz 1

$$\Sigma_1=O(1)\sum_{\nu=0}^{n}\binom{n}{\nu}\int_0^1 t^\nu(1-t)^{n-\nu}t^{-r}\,dt\,n^{k_1\delta}(\nu+1)^{-k_1\delta}=$$

$$=O(1)\sum_{\nu=0}^{n}\frac{(\nu+1)^{-r}}{(n+1)^{1-r}}n^{k_1\delta}(\nu+1)^{-k_1\delta}=O(1).$$

Nun betrachten wir Σ_2

$$\Sigma_2 = O(1) \sum_{\nu=0}^{n} \frac{n+1}{\nu+1} \binom{n}{\nu} \int_{\varepsilon}^{1} t^{\nu}(1-t)^{n-\nu} z(t)\, dt =$$

$$= O(1) \int_{\varepsilon}^{1} \sum_{\nu=0}^{n} \binom{n+1}{\nu+1} t^{\nu+1}(1-t)^{n+1-\nu-1} \frac{z(t)}{t}\, dt = O(1)$$

da $z(t) \in L[0, 1]$. Damit ist (I) bewiesen.

Zu (II). Zum Beweis von (II) benutzen wir dieselbe Aufspaltung des Integrals mit den damit verbundenen Abschätzungen wie im Beweis zu (I).

$$n^{k_1\delta}(\nu+1)^{-k_1\delta} z_{n\nu} = n^{k_1\delta}(\nu+1)^{-k_1\delta} \binom{n}{\nu} \int_{0}^{\varepsilon} t^{\nu}(1-t)^{n-\nu} z(t)\, dt +$$

$$+ n^{k_1\delta}(\nu+1)^{-k_1\delta} \binom{n}{\nu} \int_{\varepsilon}^{1} t^{\nu}(1-t)^{n-\nu} z(t)\, dt =$$

$$= O(1) \left(\frac{(\nu+1)^{-r}}{(n+1)^{1-r}} (n+1)^{k_1\delta}(\nu+1)^{-k_1\delta} + \right.$$

$$\left. + \frac{(n+1)^{k_1\delta-1}}{(\nu+1)^{k_1\delta-1}} \int_{\varepsilon}^{1} \binom{n+1}{\nu+1} t^{\nu+1}(1-t)^{n-\nu} \frac{z(t)}{t}\, dt \right) =$$

$$= O(1)\left((\nu+1)^{-r-k_1\delta}(n+1)^{r-1+k_1\delta} + (\nu+1)^{1-k_1\delta}(n+1)^{k_1\delta-1}\right) = o(1)$$

für jedes feste ν und $k_1\delta < 1-r$. Damit ist Satz 2 bewiesen.

Beweis zu Satz 1. Mit Satz IV von L. Leindler folgt aus obiger Bedingung für $0<\alpha<1$, $0<\delta<\alpha/2$ und $\beta'>-1/2$

$$\left(\frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |\sigma_{\nu}^{\beta'}(x) - f(x)|^2 \right)^{1/2} = o_x(n^{-\delta}).$$

Wir schalten vor das innere Verfahren ein reguläres Cesàro-Verfahren der Ordnung γ und können unter Verwendung von Satz 2 eine Aussage für größere k-Parameterwerte machen. Nach Definition der Matrixelemente für Cesàroverfahren gilt für die Funktionen $g(t)$ und $h(t)$ aus Satz 2:

$$g(t) = \gamma(1-t)^{\gamma-1}; \quad g(t) \in L^p[0, 1] \quad \text{mit} \quad p = \frac{1}{1-\gamma} - \varepsilon', \quad 0 < \varepsilon'$$

$$h(t) = \alpha(1-t)^{\alpha-1}$$

und somit

$$z(t) = O(1)(1-t)^{k(\gamma-1)(1-(1/2(1-\gamma))+(\varepsilon'/2))+(\alpha-1)(1-(k/2))}.$$

Es ist zu zeigen: $z(t)\in L[0, 1]$. Dazu muß gelten

$$k(\gamma-1)\big(1-(1/2(1-\gamma))+(\varepsilon'/2)\big)+(\alpha-1)\big(1-(k/2)\big)>-1.$$

Dies ist erfüllt, falls gilt

$$\gamma>\alpha\left(\frac{1}{2}-\frac{1}{k}\right).$$

Weiterhin muß gelten $z(t)=O(t^{-s})$ für $t\in[0, \varepsilon]$. Dies ist für $s>0$ erfüllt, da $z(t)$ in der Umgebung von Null beschränkt ist. Mit dem Satz von L. Leindler und Satz 2 folgt die Behauptung.

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On imbedding theorems for weighted polynomial approximation and modulus of continuity of functions

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0. Introduction

Let φ and ψ be two measurable functions on (a, b), $(-\infty \leqq a < b \leqq \infty)$. Denote $\varphi(L)\psi(L) = \varphi(L)\psi(L)_{(a,b)}$ the set of those measurable functions f on (a, b) for which

$$\int_a^b \varphi(|f(x)|)\,\psi(|f(x)|)\,dx$$

exists. In the case $\psi \equiv 1$ and $\varphi(x) = |x|^p$ $(1 \leqq p < \infty)$ we usually write L^p instead of $\varphi(L)\psi(L)$.

The norm of $f \in L^p(a, b)$ is defined by

$$\|f\|_p = \Big(\int_a^b |f(x)|^p\,dx\Big)^{1/p}.$$

The space L^p of all the functions of periodic 2π will be denoted by $L^p[2\pi]$. The modulus of continuity of a function $f \in L^p(a, b)$ is defined as follows

$$\omega(f, \delta)_p = \sup_{0 \leqq h \leqq \delta} \Big(\int_a^{b-h} |f(x+h) - f(x)|^p\,dx\Big)^{1/p} \quad (0 \leqq \delta \leqq b-a).$$

If $f \in L^p[2\pi]$ then let

$$\omega(f, \delta)_p = \sup_{0 \leqq h \leqq \delta} \Big(\int_0^{2\pi} |f(x+h) - f(x)|^p\,dx\Big)^{1/p} \quad (\delta \geqq 0).$$

A nondecreasing continuous function Ω on $[0, 1]$ is called a modulus of continuity if

$$\Omega(0) = 0, \quad \Omega(\delta_1 + \delta_2) \leqq \Omega(\delta_1) + \Omega(\delta_2) \quad (0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1 + \delta_2 \leqq 1).$$

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For a modulus of continuity Ω and $1 \leqq p < \infty$ let

$$H_p^{\Omega} = H_p^{\Omega,\omega} := \{f \in L^p \colon \omega(f, \delta) \leqq c(f)\,\Omega(\delta),\ \delta > 0\}$$

here and later $c(x, \ldots)$ denotes a constant depending only on $x, \ldots$, furthermore c will denote an absolute constant (not necessarily the same in different formulae).

Let $F = \{f_n\}_{n=0}^{\infty}$ be an orthonormal system on (a, b). Define for $n = 0, 1, \ldots$

$$\Pi_n(F) := \Big\{p_n = \sum_{k=0}^{n} \lambda_k f_k \colon \lambda_k \text{ are real numbers, } k = 0, 1, \ldots\Big\}.$$

If for some $1 \leqq p < \infty$, $F \subset L^p$, then let

$$E_n(F, f)_p := \inf_{p_n \in \Pi_n(F)} \|f - p_n\|_p \quad (f \in L^p,\ n = 0, 1, \ldots).$$

For a given decreasing sequence of real numbers tending to zero $\alpha = (\alpha_n) = (\alpha_n \downarrow 0)$, let

$$E(F, \alpha, p) := \{f \in L^p \colon E_n(F, f)_p \leqq c(f)\alpha_n,\ n = 0, 1, \ldots\}.$$

Many authors have studied the so-called imbedding problems: What are sufficient conditions and what are necessary conditions (regarding Ω) for

$$H_p^{\Omega,\omega} \subset A, \tag{1}$$

where A is a given set of functions. A similar problem is to find sufficient conditions and necessary conditions (regarding α) for

$$E(F, \alpha, p) \subset B, \tag{2}$$

where B denotes some given set of functions. For example Ul'janov [10] considered these problems in the case $A = B = L^q[2\pi]$ $(1 \leqq p < q < \infty)$ and if F is the trigonometric system. Timán [9] answering one of Ul'janov's questions proved that a certain sufficient condition due to Ul'janov is also necessary for imbedding (2) with $B = L^q[2\pi]$. L. Leindler generalized these results for $A = B = \varphi(L)\psi(L)$ (see e.g. [4], [5]). Some analogous results on the infinite interval due to J. Németh [8].

Let $\lambda > 0$. The orthonormal system F is called (by the present author) a $\{N, \lambda\}$-system if the inequality

$$\|p_n\|_q \leqq c n^{\lambda((1/p)-(1/q))} \|p_n\|_p \tag{3}$$

holds for every $p_n \in \Pi_n(F)$, $n = 1, 2, \ldots$ and $1 \leqq p < q < \infty$. In the case $\lambda = 1$, inequality (3) is called Nikol'skiĭ-inequality.

The following statement is true, its proof is similar to that of Timan [9]. Let F be a $\{N, \lambda\}$-system and let $f \in L^p$ $(1 \leqq p < \infty)$. If for some $1 \leqq p < q < \infty$

$$\varepsilon := \sum_{n=1}^{\infty} n^{\lambda(q/p-1)-1} E_n^q(F, f)_p < \infty \tag{4}$$

then $f\in L^q$ and $\|f\|_q \leqq c\{\|f\|_p^q+\varepsilon\}^{1/q}$. Consequently, for a $\{N,\lambda\}$-system, the condition

$$\sum_{n=1}^{\infty} n^{\lambda(q/p-1)-1}\alpha_n^q < \infty \tag{5}$$

is sufficient for imbedding (2) with $B=L^q$. We can ask if this is also necessary.

On the other hand, many results of the approximation theory show that for a given system F there exist new moduli of continuity for which the analogues of Jackson and Bernstein theorems are true. Therefore the following problem seems to be natural: What can we say about imbedding (1) in the case if ω is also a modulus of continuity?

In this paper we give an answer to the first question in the case of the generalized Hermite functions and we consider the second problem for the modulus of continuity to be defined later on. Some results will be proved for $\varphi(L)\psi(L)$ as well.

1. The main results

Let

$$w(x) = (1+|x|^u)^{v/2u}e^{-|x|^u/2} \quad (-\infty < x < \infty), \quad u \geqq 2, \quad v \geqq 0$$

and let $\{h_n\}$ be the system of the orthonormal polynomials with respect to the weight w^2. Then the system $F_{u,v}=\{f_n w\}$ is orthonormal on $(-\infty,\infty)$. If $u=2$, $v=0$ then $F_{u,v}$ is the system of the orthonormal Hermite functions. The weight w was introduced by FREUD [2] for all real v and $u\geqq 2$. In this paper, when no additional condition is required, we always assume that $v\geqq 0$, $u\geqq 2$.

We define the modulus of continuity of a function $f\in L^p(-\infty,\infty)$ as follows

$$\omega^*(f,\delta)_p = \omega^*_{A,B}(f,\delta)_p = \omega^*_{A,B}(u,v,f,\delta)_p = \tag{6}$$

$$= \sup_{0\leqq h\leqq\delta}\Big\{\int_{-\infty}^{B} |f_p(x+h)-f_p(x)|^p w^p(x)\,dx\Big\}^{1/p} +$$

$$+ \sup_{0<h\leqq\delta}\Big\{\int_{A}^{\infty} |f_p(x-h)-f_p(x)|^p w^p(x)\,dx\Big\}^{1/p} \quad (\delta>0,\ -\infty<A<B<\infty),$$

where

$$f_p := w^{-p} f.$$

The modulus of this type was introduced in [3].

For a given sequence of real numbers (φ_n) and $1 \leqq p, q < \infty$, let

$$\Phi(x) = \Phi_{p,q,\lambda}(x) := \sum_{k=1}^{[x]} k^{\lambda(q/p-1)-1} \varphi_k. \tag{7}$$

In the case $\lambda = 1$ this function was introduced by LEINDLER [6].

Further on we simply write $\varphi(L)\psi(L)$ for $\varphi(L)\psi(L)_{(-\infty,\infty)}$.

The following theorems are true:

Theorem 1. *Let* $1 \leqq p \leqq q < \infty$ *and let* $\alpha = (\alpha_n \downarrow 0)$, (φ_n) *be given nonnegative monotonic sequences satisfying*

$$n\alpha_n \leqq cm\alpha_m \quad \text{for} \quad 1 \leqq n < m \tag{8}$$

and $\varphi_{k^2} \leqq c\varphi_k$, *and if* $q > p$ *then let* (φ_n) *be decreasing. Let* $\Phi = \Phi_{p,q,\lambda}$ *be the function defined in* (7) *with* $\lambda = 1 - 1/u$. *Then a necessary condition for*

$$E(F_{u,v}, \alpha, p) \subset L^{q+(1-1/u)(1-q/p)} \Phi(L) \tag{9}$$

is

$$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \varphi_n \alpha_n^q < \infty. \tag{10}$$

Theorem 2. *Let* $1 \leqq p < q < \infty$ *and let* $\alpha = (\alpha_n \downarrow 0)$ *be a sequence having the properties required in Theorem* 1. *Let* $v_0 = 0$. *A necessary and sufficient condition for*

$$E(F_{u,v_0}, \alpha, p) \subset L^q \tag{11}$$

is

$$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \alpha_n^q < \infty. \tag{12}$$

Theorem 3. *Let* Ω *be a modulus of continuity,* $1 \leqq p \leqq q < \infty$ *and let* (φ_n) *be a sequence having the properties required in Theorem* 1. *Let* $\Phi = \Phi_{p,q,\lambda}$ *with* $\lambda = 1 - 1/u$. *A necessary condition for*

$$H_p^{\Omega,\omega^*} \subset L^{q+(1-1/u)(1-q/p)} \Phi(L) \tag{13}$$

is

$$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \varphi_n \Omega^q(n^{-(1-1/u)}) < \infty. \tag{14}$$

Theorem 4. *Let* $1 \leqq p < q < \infty$ *and let* $\omega_0^* = \omega_{A,B}(u, v_0, f, \delta)_p$ *with* $v_0 = 0$. *A necessary and sufficient condition for*

$$H_p^{\Omega,\omega_0^*} \subset L^q \tag{15}$$

is

$$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1} \Omega^q(n^{-(1-1/u)}) < \infty. \tag{16}$$

2. Lemmas

Lemma 1 ([6], Lemma 5). *Let* $p>0$ *and let* $(\alpha_n\downarrow 0)$ *be a sequence satisfying* (8). *Let* (φ_n) *be a nonnegative monotonic sequence having the property that for a certain* α

$$\sum_{k=m}^{\infty}\frac{\varphi_k}{k^{\alpha+p}}\leqq c\,\frac{m\varphi_m}{m^{\alpha+p}} \tag{17}$$

and

$$\sum_{k=1}^{\infty}\varphi_k k^{-\alpha}\alpha_k^p=\infty. \tag{18}$$

Then there exists a sequence $\{B_k\}$ *such that*

$$B_k\downarrow 0,\quad B_k\leqq\alpha_k,\quad \sum_{k=1}^{m}k^{\lambda p-1}B_k^p\leqq c(\lambda,p)\,m^{\lambda p}\alpha_m^p\quad \textit{for any}\quad \lambda>0 \tag{19}$$

and

$$\sum_{k=1}^{\infty}\varphi_k k^{-\alpha}B_k^p=\infty. \tag{20}$$

This lemma differs from Lemma 5 of [6] in the rate of λ, since the last inequality in (19) is true for any $\lambda>0$ (in [6] this inequality was proved for $\lambda=1$). Indeed, the sequence $\{B_k\}$ defined in [6] has property (19). The proof of this fact is similar to that of the last inequality in (2.4) of [6].

We have similar remark concerning the inequality (25) in the following lemma.

Lemma 2 ([6], Lemma 6). *Let* $p\geqq 1$, $\alpha<1$ *and let* $(\alpha_n\downarrow 0)$ *be a sequence satisfying* (8). *If for the positive increasing sequence* (φ_n),

$$\sum_{k=m}^{\infty}\varphi_k k^{-\alpha-p}\leqq c\varphi_m m^{1-\alpha-p} \tag{21}$$

and

$$\sum_{n=1}^{\infty}\varphi_n n^{-\alpha}\alpha_n^p=\infty \tag{22}$$

hold, then there exist a sequence $\{B_k\}$ *and a sequence of integers* $\{n_k\}$ *such that*

$$B_n\downarrow 0,\quad B_n\leqq\alpha_n \tag{23}$$

$$n_{k+1}>2n_k\quad \textit{and}\quad B_{n_{k+1}}\leqq\frac{1}{2}B_{n_k}\quad (k\geqq 1) \tag{24}$$

$$\sum_{n=1}^{m}n^{\lambda q-1}B_n^q\leqq c(q,\lambda)\,m^{\lambda q}\alpha_m^q\quad \textit{for any}\quad q,\lambda>0 \tag{25}$$

$$\sum_{n=1}^{\infty}\varphi_n n^{-\alpha}B_n^p=\infty\quad \textit{and}\quad \sum_{k=1}^{\infty}\varphi_{n_k}n_k^{1-\alpha}B_{n_k}=\infty \tag{26}$$

and

$$\sum_{k=1}^{\infty} \varphi_{2^n} 2^{n(1-\alpha)} (B_{2^n} - B_{2^{n-1}})^p = \infty. \tag{27}$$

Lemma 3. *Let* $1 \leqq p \leqq q < \infty$, $\lambda \geqq 1/2$ *and let* (α_n), (φ_n) *be sequences having the properties required in Theorem* 1. *If*

$$\sum_{n=1}^{\infty} n^{\lambda(q/p-1)-1} \varphi_n \alpha_n^q = \infty \tag{28}$$

then there exists a function $f_0 \in L^p[0, 1]$ *having the following properties:*

$$f_0(x) = 0, \quad x \in [2^{-\lambda}, 1] \tag{29}$$

$$\int_0^h |f_0(x)|^p \, dx \leqq c\alpha_{2^k}^p \quad (0 < h \leqq 2^{-\lambda(k+2)}, \ k = 1, 2, \ldots) \tag{30}$$

$$\omega(f_0, 2^{-\lambda k})_p \leqq c\alpha_{2^k}, \quad k = 1, 2, \ldots \tag{31}$$

and

$$f_0 \notin L^{q+\lambda(1-q/p)} \Phi(L), \tag{32}$$

where $\Phi = \Phi_{p,q,\lambda}$ *is defined by* (7).

Proof. First we remark that in the case $\lambda = 1$ this lemma was proved in [6], [7]. Here we use a similar method for the construction of f_0.

If $q = p$ then the conditions of Lemma 1 with $\alpha = 1$ are satisfied, so there exists a sequence $\{\bar{B}_k\}$ satisfying (19) and (20) with $\alpha = 1$.

If $q > p$ then the conditions of Lemma 2 are satisfied with $\alpha = 1 - \lambda\left(\frac{q}{p} - 1\right)$ and the exponent p appearing in Lemma 2 is chosen to be q. Therefore there exist $\{\hat{B}_k\}$ and $\{n_k\}$ satisfying (23)—(27).

Now we can define

$$f_0(x) = \begin{cases} \varrho_n & \text{if} \quad x = 3^\lambda 2^{-\lambda(n+2)} \\ 0 & \text{if} \quad x \in [2^{-\lambda}, 1], \quad x = 2^{-\lambda n} \\ \text{linear} & \text{on} \ [2^{-\lambda(n+1)}, 3^\lambda 2^{-\lambda(n+2)}], \\ & \qquad [3^\lambda 2^{-\lambda(n+2)}, 2^{-\lambda n}], \quad n = 1, 2, \ldots, \end{cases} \tag{33}$$

where

$$\varrho_n := 2^{(n+1)\lambda/p} (B_{2^n}^p - B_{2^{n+1}}^p)^{1/p} \quad (n = 1, 2, \ldots) \tag{34}$$

with

$$B_n := \begin{cases} \bar{B}_n & \text{if} \quad p = q \\ \hat{B}_n & \text{if} \quad p < q. \end{cases}$$

Let

$$\sigma := \frac{\sqrt{2}-1}{2} \tag{35}$$

and let

$$h \in (\sigma 2^{-\lambda(k+3)}, \sigma 2^{-\lambda(k+2)}], \quad k \geqq 2. \tag{36}$$

Then it is easy to see that

$$0 < (4^\lambda - 1)h < 1-h.$$

We have

$$\int_0^{1-h} |f_0(t+h)-f_0(t)|^p\,dt = \int_0^{(4^\lambda-1)h} + \int_{(4^\lambda-1)h}^{1-h} =: I_1 + I_2.$$

By (19) and (23) we get

$$I_1 \leqq \int_0^{4^\lambda h} |f_0(x)|^p\,dx \leqq \int_0^{2^{-\lambda k}} |f_0(x)|^p\,dx = \sum_{n=k}^{\infty} \int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}} |f_0(x)|^p\,dx =$$

$$= (1-2^{-\lambda}) \sum_{n=k}^{\infty} 2^{-\lambda(n+1)} \varrho_n^p = (1-2^{-\lambda}) \sum_{n=k}^{\infty} (B_p^{2^n} - B_{2^{n+1}}^p) \leqq$$

$$\leqq c(\lambda) B_{2^k}^p \leqq c(\lambda) \alpha_{2^k}^p.$$

To estimate I_2 we notice that by (35) and (36) we have for every $t \leqq 2^{-\lambda n}$, $1 \leqq n \leqq$ $\leqq k+1$

$$t+h \leqq 2^{-\lambda(n-1)}.$$

Therefore for those values of t

$$|f_0(t+h)-f_0(t)| \leqq (\varrho_n + \varrho_{n-1}) \leqq ch 2^{\lambda(n+2)}(\varrho_n+\varrho_{n-1}).$$

Now, using (19), (23) and (25) we have

$$I_2 = \int_{(4^\lambda-1)h}^{1-h} |f_0(t+h)-f_0(t)|^p\,dt \leqq \int_{2^{-\lambda(k+2)}}^{2^{-\lambda}} |f_0(t+h)-f_0(t)|^p\,dt =$$

$$= \sum_{n=1}^{k+1} \int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}} |f_0(t+h)-f_0(t)|^p\,dt \leqq c \sum_{n=1}^{k+1} 2^{-\lambda n}[h 2^{\lambda(n+2)}(\varrho_n+\varrho_{n-1})]^p \leqq$$

$$\leqq ch^p \sum_{n=1}^{k+1} 2^{\lambda(p-1)n}(\varrho_n+\varrho_{n-1})^p \leqq ch^p \sum_{n=0}^{k+1} 2^{\lambda(p-1)n} \varrho_n^p =$$

$$= ch^p \sum_{n=0}^{k+1} 2^{\lambda pn}(B_{2^n}^p - B_{2^{n+1}}^p) \leqq ch^p \sum_{n=0}^{k+1} 2^{\lambda pn} B_{2^n}^p \leqq$$

$$\leqq ch^p \sum_{i=1}^{2^{k+1}} i^{\lambda p-1} B_i^p \leqq ch^p (2^{k+1})^{\lambda p} \alpha_{2^{k+1}}^p \leqq c\alpha_{2^k}^p.$$

So $I_1+I_2\leqq c\alpha_{2^k}^p$, from which (31) follows. (29) follows from the definition of f_0. We obtain (30) by the estimate of I_1.

Now let us prove (32). If $q=p$ then the function $\Phi_{p,q,\lambda}$ and the sequence $\{B_n\}$ do not depend on λ, therefore we can use the estimates on p. 61 of [6]. According to this, for $N=1, 2, \dots$, there exists μ depending on N such that

$$\sum_{k=1}^{N}\varphi_k k^{-1}B_k^p \leqq c\sum_{n=1}^{\mu-1}\Phi(2^n)(B_{2^n}^p-B_{2^{n+1}}^p)+c\leqq \tag{37}$$

$$\leqq c\sum_{k=1}^{\mu}\Phi(2^n)\varrho_n^p 2^{-\lambda(n+1)}+c.$$

Since by our assumption and (20) the first sum in inequality (37) tends to infinity as $N\to\infty$, therefore

$$\sum_{n=1}^{\mu}\Phi(2^n)\varrho_n^p 2^{-\lambda n}\to\infty \quad (\mu\to\infty). \tag{38}$$

On the other hand, by the assumption $\varphi_{k^2}\leqq c\varphi_k$ we have $\Phi(u^2)\leqq c\Phi(u)$. Consequently, since $\lambda\geqq 1/2$ we get

$$\Phi(2^n)\leqq c\Phi(2^{\lambda n}).$$

Hence by (38) we have

$$\sum_{n=1}^{\infty}\Phi(2^{\lambda n})\varrho_n^p 2^{-\lambda n}=\infty. \tag{39}$$

However,

$$\int_0^1 |f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx=\sum_{n=0}^{\infty}\int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}}|f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx\geqq$$

$$\geqq\sum_{n=0}^{\infty}\Phi(2^{\lambda n})\int_{2^{-\lambda(n+1)}}^{2^{-\lambda n}}|f_0(x)|^p\,dx\geqq c\sum_{n=0}^{\infty}\Phi(2^{\lambda n})\varrho_n^p 2^{-\lambda n}.$$

So by (39)

$$\int_0^1 |f_0(x)|^p\Phi\left(\frac{1}{x}\right)dx=\infty. \tag{40}$$

This together with the property $\Phi(u^2)\leqq c\Phi(u)$ implies by Lemma 13 of [10] that $f_0\notin L^p\Phi(L)_{[0,1]}$, which proves (32) for $q=p$.

Let now $q>p$. Using the assumption $\varphi_{k^2}\leqq c\varphi_k$ we have

$$\Phi(x)=\sum_{k=1}^{[x]}k^{\lambda(q/p-1)-1}\varphi_k\geqq\sum_{k=[x/2]}^{[x]}k^{\lambda(q/p-1)-1}\varphi_k\geqq cx^{\lambda(q/p-1)}\varphi_{[x]}.$$

Therefore

$$\int_0^1 |f_0(x)|^{q+\lambda(1-q/p)}\Phi(|f_0(x)|)\,dx \geqq c\int_0^1 |f_0(x)|^q \varphi(|f_0(x)|)\,dx, \tag{41}$$

where

$$\varphi(x) = \begin{cases} 0 & \text{if } x\in[0, 1] \\ \varphi_n & \text{if } x = n \\ \text{linear on} & [n, n+1], \quad n = 1, 2, \dots \end{cases}$$

Using Lemma 2 with $\alpha=1-\lambda(q/p-1)$ we have

$$\sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\lambda(q/p-1)}(B_{2^n}-B_{2^{n+1}})^q = \infty. \tag{42}$$

On the other hand

$$\int_0^1 |f_0(x)|^q \varphi\left(\frac{1}{x}\right) dx = \sum_{n=1}^{\infty} \int_{2^{-n\lambda}}^{2^{-(n-1)\lambda}} |f_0(x)|^q \varphi\left(\frac{1}{x}\right) dx \geqq$$

$$\geqq c(\lambda)\sum_{n=1}^{\infty} \varphi(2^{n\lambda})\varrho_n^q 2^{-n\lambda} \geqq c(\lambda)\sum_{n=1}^{\infty} \varphi_{2^n}\varrho_n 2^{-n\lambda} \geqq$$

$$\geqq c(\lambda)\sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\lambda(q/p-1)}(B_{2^n}^p - B_{2^{n+1}}^p)^{q/p} \geqq c(\lambda)\sum_{n=1}^{\infty} \varphi_{2^n} 2^{n\lambda(q/p-1)}(B_{2^n}-B_{2^{n+1}})^q.$$

Hence by (42) we get

$$\int_0^1 |f_0(x)|^q \varphi\left(\frac{1}{x}\right) dx = \infty.$$

Therefore again using Lemma 13 of [10] we have

$$\int_0^1 |f_0(x)|^q \varphi(|f_0(x)|)\,dx = \infty$$

so, by (41)

$$f_0 \notin L^{q+\lambda(1-q/p)}\Phi(L).$$

This completes the proof of Lemma 3.

Lemma 4 ([7], Theorem 3.1). *Let* $v_0=0$, $u\geqq 2$, $n=1, 2, \dots$. *Then for any* $p_n\in\Pi_n(F_{u,v_0})$ *and* $1\leqq p<q<\infty$ *we have*

$$\|p_n\|_q \leqq c n^{(1-1/u)(1/p-1/q)}\|p_n\|_p. \tag{43}$$

Lemma 5 ([1], Lemma 3.6 and [2] Lemma 4.7). *Let* $1 \leqq p < \infty$. *Suppose that a function g is absolutely continuous on every finite interval and* $f := wg$, $wg' \in L^p$, *then*

(44) $$E_n(F_{u,v}, f)_p \leqq \frac{c}{n^{1-1/u}} \|wg'\|_p \quad (n = 1, 2, \ldots).$$

Lemma 6. *Let* $1 \leqq p < \infty$. *For any* $f \in L^p$ *and* $-\infty < A < B < \infty$ *we have*

(45) $$E_n(F_{u,v}, f)_p \leqq c(A, B)\, \omega^*_{A,B}(f, n^{-(1-1/u)})_p \quad (n = 1, 2, \ldots)$$

where ω^* *is defined in* (5).

Proof. The existence of ω^* indeed follows from the following inequalities

(46) $$\begin{cases} w(x) \leqq c(B, \delta)\, w(x+h) & (-\infty < x \leqq B) \\ w(x) \leqq c(A, \delta)\, w(x-h) & (A \leqq x < \infty) \end{cases} \quad (\delta > 0,\ 0 < h \leqq \delta).$$

Let now

$$\lambda_n := n^{-(1-1/u)}, \quad f_p := w^{-p} f.$$

By Minkowskii-inequality we have

$$\Big\{\int_A^B \Big|2\lambda_n^{-1} \int_{\lambda_n/2}^{\lambda_n} [f_p(x+t) - f_p(x-t)]\, dt\Big|^p dx\Big\}^{1/p} \leqq$$

$$\leqq 2c\lambda_n^{-1} \int_{\lambda_n/2}^{\lambda_n} \Big\{\int_A^B |f_p(x+t) - f_p(x-t)|^p w^p(x)\, dx\Big\}^{1/p} dt \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p.$$

Hence, it follows that there exists an $A \leqq x_n \leqq B$ such that

$$|d_n| \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p$$

with

$$d_n := 2\lambda_n^{-1} \int_{\lambda_n/2}^{\lambda_n} [f_p(x_n+t) - f_p(x_n-t)]\, dt.$$

Let

$$\varphi_n(x) := \begin{cases} 2\lambda_n^{-1} \displaystyle\int_{\lambda_n/2}^{\lambda_n} f_p(x+t)\, dt & \text{if} \quad x \leqq x_n \\ 2\lambda_n^{-1} \displaystyle\int_{\lambda_n/2}^{\lambda_n} f_p(x-t)\, dt + d_n & \text{if} \quad x > x_n. \end{cases}$$

Then it is easy to see that

$$\|(f_p - \varphi_n) w\|_p \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p$$

and

$$\|\varphi_n' w\|_p \leqq c(A, B)\, \omega^*_{A,B}(f, \lambda_n)_p \cdot \lambda_n^{-1}.$$

Since φ_n is absolutely continuous on every finite interval, by the last two inequalities, using Lemma 6 we get

$$E_n(F_{u,v},f)_p \leqq \|(f_p-\varphi_n)w\|_p+E_n(F_{u,v},\varphi_n w) \leqq$$
$$\leqq c(A,B)\,\omega^*_{A,B}(f,\lambda_n)_p+c(A,B)\lambda_n\|\varphi_n' w\|_p \leqq$$
$$\leqq c(A,B)\,\omega^*_{A,B}(f,\lambda_n)_p,$$

which proves (45).

4. Proofs of the theorems

Proof of Theorem 1. Suppose that

$$\sum_{n=1}^{\infty} n^{(1-1/u)(q/p-1)-1}\varphi_n \alpha_n^q = \infty. \tag{47}$$

Then by Lemma 3 there exists a function $f_0 \in L^p[0,1]$ satisfying (29)—(32) with $\lambda = 1-\frac{1}{u}\left(\geqq\frac{1}{2}\right)$.

We define

$$f(x)=\begin{cases} d f_0(x)\,w^p(x) & \text{if} \quad x\in[0,1] \quad \text{with} \quad d=e^{p/2},\\ 0, & \text{if} \quad x\notin[0,1],\end{cases}$$

and estimate $\omega^*_{A,B}(f,\delta)_p$ with $A=2, B=3$. By $\lambda:=1-\dfrac{1}{u}$ we have for $1-h \geqq 2^{-\lambda}$

$$I_1(h):=\int_{-\infty}^{3}|f_p(x+h)-f_p(x)|^p w^p(x)\,dx \leqq$$
$$\leqq c\int_0^h |f_0(x)|^p\,dx + c\int_0^{1-h}|f_0(x+h)-f_0(x)|^p\,dx.$$

Hence by (30), (31) we get

$$I_1(h) \leqq c\alpha_{2^k}^p \quad \text{if} \quad h \leqq 2^{-\lambda(k+2)}, \quad k=1,2,\ldots.$$

Therefore by the definition of ω^* we have

$$\omega^*_{2,3}(f,2^{-\lambda k})_p \leqq c\alpha_{2^k} \quad (k=1,2,\ldots),$$

from which it follows by (45) that

$$E_{2^k}(F_{u,v},f)_p \leqq c\alpha_{2^k} \quad (k=1,2,\ldots).$$

Since $n\alpha_n \leqq cm\alpha_m$ for $1 \leqq n < m$, we obtain

$$E_n(F_{u,v}, f)_p \leqq c\alpha_n \quad n = 1, 2, \ldots),$$

too. This proves that $f \in E(F_{u,v}, \alpha, p)$.

On the other hand, since $f(x) \geqq f_0(x)$ $(x \in [0, 1])$, by (32) we have

$$f \notin L^{q+\lambda(1-q/p)} \Phi(L).$$

The proof of Theorem 1 is completed.

Remark 1. In the proof of Theorem 1 the chosen values of constants A and B in ω^* indeed are not essential. For any $-\infty < A < B < \infty$ by similar method we can construct a function f such that $f \notin L^{q+\lambda(1-q/p)} \Phi(L)$ and

$$\omega^*_{A,B}(f, 2^{-\lambda k})_p \leqq c\alpha_{2^k} \quad (k = 1, 2, \ldots).$$

Proof of Theorem 2. If $\varphi_n = 1$ $(n = 0, 1, \ldots)$ then

$$L^{q+\lambda(1-q/p)} \Phi_{p,q,\lambda}(L) = L^q.$$

Therefore the necessary part in Theorem 2 follows from Theorem 1. The sufficient part is a consequence of the statement summarized in the introduction, since by (43) F_{u,v_0} is a $\left\{N, \left(1 - \frac{1}{u}\right)\right\}$-system.

Proof of Theorem 3. Assume that series (14) is divergent. Then by virtue of Remark 1, with $\alpha_n := \Omega(n^{-(1-1/u)})$, we can construct a function $f \in L^p$ such that $f \notin L^{q+\lambda(1-q/p)} \Phi(L)$ and

$$\omega^*_{A,B}(f, 2^{-(1-1/u)k})_p \leqq c\Omega(2^{-(1-1/u)k}) \quad (k = 1, 2, \ldots).$$

Hence by the properties of the modulus of continuity it follows that

$$\omega^*_{A,B}(f, \delta)_p \leqq c\Omega(\delta) \quad (\delta > 0).$$

So, we have $f \in H_p^{\Omega, \omega^*}$.

Proof of Theorem 4. The necessary part of Theorem 4 is a consequence of Theorem 3. The sufficient part follows from Theorem 2 and Lemma 6.

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On Young-type inequalities

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1. Introduction

The present paper is inspired by the following result of LOSONCZI [10].

Let f, g: $]0, \infty[\to \mathbf{R}$ be arbitrary functions. The Young-type inequality

$$xy \leqq f(x)+g(y), \quad x, y>0 \tag{1}$$

is satisfied if and only if there exist nonnegative functions p, q: $]0, \infty[\to[0, \infty[$, a constant $\alpha \in \mathbf{R}$ and a Young function φ such that

$$f(x)=\int_0^x \varphi(t)\, dt+p(x)+\alpha, \quad x>0,$$

$$g(y)=\int_0^y \varphi^{(-1)}(s)\, ds+q(y)-\alpha, \quad y>0,$$

where $\varphi^{(-1)}$ denotes the right inverse of φ.

Here φ: $[0, \infty[\to[0, \infty[$ is called a *Young function* if it is increasing and right continuous and $\lim_{x \to \infty} \varphi(x)=\infty$. The *right inverse* of φ is defined by

$$\varphi^{(-1)}(y)= \begin{cases}0, & \text { if } \quad 0 \leqq y<\varphi(0) \\ \sup \{x \geqq 0 \mid \varphi(x) \leqq y\}, & \text { if } \quad \varphi(0) \leqq y,\end{cases}$$

and it turns out that $\varphi^{(-1)}$ is also a Young function.

Taking $\alpha=0, p \equiv q \equiv 0$, the "if" part of the above statement reduces to

$$xy \leqq \int_0^x \varphi(t)\, dt+\int_0^y \varphi^{(-1)}(s)\, ds, \quad x, y>0 \tag{2}$$

which is called Young's inequality although YOUNG [15] proved it only when the

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derivatives of φ and $\varphi^{(-1)}$ exist everywhere. There are several generalizations of this inequality. Here we mention only papers of Birnbaum and Orlicz [1], Boas and Marcus [2], [3], [4], Cooper [5], Cunningham and Grossman [6], Dankert and König [7], Diaz and Metcalf [8], Klambauer [9], Oppenheim [12] and Zaanen [15].

The "only if" part of the above result of Losonczi states that (1) can always be obtained by weakening a Young's inequality, in other words, this means that the Young inequalities are the only essential inequalities of the form (1).

In what follows, we deal with the functional inequality

$$H(x, y) \leqq f(x)+g(y), \quad a \leqq x \leqq A, \quad b \leqq y \leqq B,$$

where H is a given function and f, g are unknown functions. For a large class of functions H we prove an analogue of the theorem of Losonczi. The only point where our results are not more general than that of Losonczi [10] is that we assume x and y to be in the closed intervals $[a, A]$ and $[b, B]$, respectively.

2. Young functions

Let $[a, A]$ and $[b, B]$ be given fixed intervals throughout this paper. A function φ: $[a, A]\to[b, B]$ is called a *Young function* (cf. Losonczi [10], [11], Cunningham and Grossman [6]) if

(i) φ is increasing and right continuous,

(ii) $\varphi(A)=B$.

The *right inverse* of φ, is the function $\varphi^{(-1)}$: $[b, B]\to[a, A]$ defined by

$$\varphi^{(-1)}(y) = \begin{cases} a, & \text{if} \quad b \leqq y < \varphi(a) \\ \sup\{x \geqq a | \varphi(x) \leqq y\}, & \text{if} \quad y \geqq \varphi(a). \end{cases}$$

It is easy to see that $\varphi^{(-1)}$ is also a Young function (i.e., it is increasing, right continuous and $\varphi^{(-1)}(B)=A$), furthermore, the right inverse of $\varphi^{(-1)}$ equals φ.

A Young function φ: $[a, A]\to[b, B]$ is called *elementary* if there exist $a=:x_0<<x_1<\ldots<x_{n-1}<x_n:=A$ and $b\leqq y_1<\ldots<y_n\leqq B$ such that

$$\varphi(x) = y_i \quad \text{if} \quad x_{i-1} \leqq x < x_i \quad (i = 1, \ldots, n).$$

Then the right inverse of φ is also an elementary Young function and

$$\varphi^{(-1)}(y) = \begin{cases} a, & \text{if} \quad b \leqq y < y_1, \\ x_i, & \text{if} \quad y_i \leqq y < y_{i+1} \quad (i = 1, \ldots, n-1), \\ A, & \text{if} \quad y_n \leqq y \leqq B. \end{cases}$$

We shall need the following

Lemma. *Let* $\varphi\colon [a, A]\to[b, B]$ *be an arbitrary Young function. Then there exists an increasing sequence of elementary Young functions* $\varphi_n\colon [a, A]\to[b, B]$, $(n\in\mathbf{N})$ *satisfying*

$$\lim_{n\to\infty}\varphi_n=\varphi \quad\text{and}\quad \lim_{n\to\infty}\varphi_n^{(-1)}=\varphi^{(-1)}. \tag{3}$$

Proof. Write $\alpha:=A-a$ and let $\{\tau_1, \tau_2, \dots\}$ be the set of all points in $[a, A]$ where φ is not continuous. Denote by X_n $(n\in\mathbf{N})$ the set

$$\left\{\tau_1, \dots, \tau_n, a, a+\frac{\alpha}{2^n}, \dots, a+\frac{\alpha(2^n-1)}{2^n}, A\right\}.$$

Assume that the elements of X_n are $a=x_0<\dots<x_m=A$. Then define $\varphi_n\colon [a, A]\to\to[b, B]$ in the following way

$$\varphi_n(x)=\begin{cases}\varphi(x_i), & \text{if } x_i\leqq x<x_{i+1}, \quad i=0, \dots, m-1,\\ B, & \text{if } x=A.\end{cases}$$

Since $X_n\subseteqq X_{n+1}$ $(n\in\mathbf{N})$, it is obvious that (φ_n) is an increasing sequence of elementary Young functions.

To prove the first equality in (3), let $a\leqq x\leqq A$ be arbitrary. If either $x=a$ or $x=A$ then $\varphi_n(x)=\varphi(x)$ therefore there is nothing to show, so we assume that $a<x<A$. If φ is not continuous at x then there exists a k such that $x=\tau_k$, i.e., $x\in X_k\cap X_{k+1}\cap\dots$. Thus $\varphi_n(x)=\varphi(x)$ if $n\geqq k$. Therefore the first equality in (3) is obvious again.

Now suppose that $a<x<A$ and that φ is continuous at x. Let $\varepsilon>0$ be arbitrary. Then there exists a $\delta>0$ such that $|x-x'|<\delta$ implies $|\varphi(x)-\varphi(x')|<\varepsilon$. Therefore, if $\alpha/2^n<\delta$, there exist two consecutive elements x' and x'' of X_n such that $x-\delta<x'\leqq x<x''$. Then $\varphi_n(x)=\varphi(x')$ and $|\varphi(x')-\varphi(x)|<\varepsilon$. Thus we have proved that $|\varphi_n(x)-\varphi(x)|<\varepsilon$ if $\alpha/2^n<\delta$, i.e., the first equality in (3) holds true in this case, too.

To prove the second equality, let $b\leqq y\leqq B$ be fixed. The inequality $\varphi_n\leqq\varphi$ yields $\varphi^{(-1)}\leqq\varphi_n^{(-1)}$. Therefore, if $\varphi^{(-1)}(y)=A$ then $\varphi_n^{(-1)}(y)=A$. Thus we may assume that $x:=\varphi^{(-1)}(y)<A$. Let $n\in\mathbf{N}$ be fixed. Then there exist two consecutive elements x' and x'' of X_n such that $x'\leqq x<x''$. If $\varphi(x'')$ were equal to $\varphi(x)$, then $\varphi^{(-1)}(y)$ would be greater than or equal to x''. Thus necessarily $\varphi(x')\leqq\varphi(x)==y<\varphi(x'')$. This yields $\varphi_n(x)=\varphi(x')<\varphi(x'')=\varphi_n(x'')$. Now, by definition, $\varphi_n^{(-1)}(y)<x''$. Since $x''-x\leqq x''-x'<\alpha/2^n$, therefore

$$\varphi_n^{(-1)}(y)<x+\alpha/2^n=\varphi^{(-1)}(y)+\alpha/2^n.$$

On the other hand we have $\varphi^{(-1)}\leqq\varphi_n^{(-1)}$, thus $|\varphi_n^{(-1)}(y)-\varphi^{(-1)}(y)|<\alpha/2^n$ holds for all $n\in\mathbf{N}$. This relation shows that $\varphi_n^{(-1)}(y)$ converges to $\varphi^{(-1)}(y)$ if $\varphi^{(-1)}(y)\neq A$.

3. Generalizations of Young's inequality

Denote by $\mathscr{H}$ the set of functions H: $[a, A]\times[b, B]$ that satisfy the inequality

$$H(x, v)+H(u, y) \leqq H(x, y)+H(u, v) \tag{4}$$

for all $a\leqq x\leqq u\leqq A$, $b\leqq y\leqq v\leqq B$. We note that if H is a C^2 function, then (4) holds if and only if

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}H(x, y) \geqq 0 \tag{5}$$

is valid for all $a\leqq x\leqq A$, $b\leqq y\leqq B$.

The following theorem gives a Young-type inequality for elementary Young functions.

Theorem 1. *Let $H\in\mathscr{H}$ and assume that H is absolutely continuous on the boundary of $[a, A]\times[b, B]$. Then H is absolutely continuous in both variables, furthermore*

$$H(x, y) \leqq H(a, b)+\int_a^x \partial_1 H(t, \varphi(t))\, dt+\int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds \tag{6}$$

holds for all elementary Young functions φ: $[a, A]\to[b, B]$ and for all $a\leqq x\leqq A$, $b\leqq y\leqq B$.

(Here $\partial_1 H$ and $\partial_2 H$ denote the partial derivatives of H with respect to the first or second variable respectively.)

Proof. Let $b\leqq y\leqq B$ be fixed. We show that $x\mapsto H(x,y)$, $a\leqq x\leqq A$ is an absolutely continuous function. Since H satisfies (4), we have

$$H(u, b)-H(x, b) \leqq H(u, y)-H(x, y) \leqq H(u, B)-H(x, B)$$

for $a\leqq x\leqq u\leqq A$. Thus we obtain

$$|H(u, y)-H(x, y)| \leqq \max\{|H(u, b)-H(x, b)|,\ |H(u, B)-H(x, B)|\} \tag{7}$$

for all $x, u\in[a, A]$. By assumption,

$$x\mapsto H(x, b) \quad\text{and}\quad x\mapsto H(x, B)$$

are absolutely continuous on $[a, A]$. Therefore, by the estimate (7), the function $x\mapsto H(x,y)$ is also absolutely continuous. Thus the partial derivative $\partial_1 H(x,y)$ exists for almost all $a\leqq x\leqq A$ (if y is fixed). (See B. Sz.-Nagy [14] for the properties of absolutely continuous functions.) A similar argument shows that $y\mapsto H(x,y)$ is also an absolutely continuous function on $[b, B]$ for each fixed $a\leqq x\leqq A$.

Let $\varphi\colon [a, A]\to[b, B]$ be an arbitrary elementary Young function. Then there exist $a=x_0<x_1<\ldots<x_n=A$ and $b\leqq y_1<\ldots<y_n\leqq B$ such that

$$\varphi(t) = y_i \quad \text{if} \quad x_{i-1}\leqq t<x_i.$$

Assume that $x_{k-1}\leqq x\leqq x_k$. Then

$$\int_a^x \partial_1 H(t, \varphi(t))\,dt = \sum_{i=1}^{k-1}\int_{x_{i-1}}^{x_i}\partial_1 H(t, \varphi(t))\,dt+\int_{x_{k-1}}^{x}\partial_1 H(t, \varphi(t))\,dt =$$

$$= \sum_{i=1}^{k-1}\int_{x_{i-1}}^{x_i}\partial_1 H(t, y_i)\,dt+\int_{x_{k-1}}^{x}\partial_1 H(t, y_k)\,dt =$$

$$= \sum_{i=1}^{k-1}\big(H(x_i, y_i)-H(x_{i-1}, y_i)\big)+H(x, y_k)-H(x_{k-1}, y_k).$$

For the sake of simplicity, we write $y_0:=b$ and $y_{n+1}:=B$. Then

$$\varphi^{(-1)}(s) = x_j \quad \text{if} \quad y_j\leqq s<y_{j+1} \quad (j=0, \ldots, n).$$

Assume that $y_m\leqq y\leqq y_{m+1}$. Then

$$\int_0^y \partial_2 H\big(\varphi^{(-1)}(s), s\big)\,ds = \sum_{j=0}^{m-1}\int_{y_j}^{y_{j+1}}\partial_2 H\big(\varphi^{(-1)}(s), s\big)\,ds+\int_{y_m}^{y}\partial_2 H\big(\varphi^{(-1)}(s), s\big)\,ds =$$

$$= \sum_{j=0}^{m-1}\int_{y_j}^{y_{j+1}}\partial_2 H(x_j, s)\,ds+\int_{y_m}^{y}\partial_2 H(x_m, s)\,ds =$$

$$= \sum_{j=0}^{m-1}\big(H(x_j, y_{j+1})-H(x_j, y_j)\big)+H(x_m, y)-H(x_m, y_m).$$

To prove (6), we distinguish two cases.

If $k\leqq m$, then

$$\Delta := \int_a^x \partial_1 H\big(t, \varphi(t)\big)\,dt+\int_b^y \partial_2 H\big(\varphi^{(-1)}(s), s\big)\,ds+H(a, b)-H(x, y) =$$

$$= \big(H(x_k, y)+H(x, y_k)-H(x, y)-H(x_k, y_k)\big)+$$

$$+\sum_{i=k}^{m-1}\big(H(x_{i+1}, y)+H(x_i, y_{i+1})-H(x_i, y)-H(x_{i+1}, y_{i+1})\big).$$

Now applying (4), one can check that all the terms on the right-hand side of this equation are nonnegative. Thus (6) is valid in this case.

If $m<k$, then

$$\Delta = \big(H(x, y_k)+H(x_{k-1}, y)-H(x_{k-1}, y_k)-H(x, y)\big)+$$
$$+\sum_{i=m}^{k-2}\big(h(x_{i+1}, y_{i+1})+H(x_i, y)-H(x_i, y_{i+1})-H(x_{i+1}, y)\big)$$

and a similar argument shows that $\Delta \geqq 0$ is also satisfied. Thus (6) is proved in both cases.

Remark. One may ask whether (6) is true for all Young functions $\varphi\colon [a, A]\to[b, B]$ under the regularity assumptions of the theorem. The following example shows that it is not so: Let $H(x,y)=\min(x,y)$ and $\varphi(x)=x$ for all $x, y\in[0, 1]$. Then $H\in\mathscr{H}$ and H is absolutely continuous on the boundary of $[0, 1]\times[0, 1]$. However, the values $\partial_1 H(t, \varphi(t))$ and $\partial_2 H(\varphi^{(-1)}(s), s)$ are not defined for any $t, s\in[0, 1]$. Thus the right-hand side of (6) has no meaning. Therefore in order to prove (6) for arbitrary Young functions φ, we need stronger regularity properties of H.

Theorem 2. *Let $H\in\mathscr{H}$ and assume that the partial derivatives $\partial_1 H(x, y)$ and $\partial_2 H(x, y)$ exist for all $a\leqq x\leqq A$, $b\leqq y\leqq B$, furthermore*

$$y\mapsto\partial_1 H(x, y), \quad (b\leqq y\leqq B) \quad \text{and} \quad x\mapsto\partial_2 H(x, y), \quad (a\leqq x\leqq A)$$

are continuous functions for almost all fixed $a\leqq x\leqq A$ and $b\leqq y\leqq B$, respectively. Then (6) is satisfied for all Young functions $\varphi\colon [a, A]\to[b, B]$ and $a\leqq x\leqq A$, $b\leqq y\leqq B$.

Proof. Since H satisfies (4), we have

$$\frac{H(u, y)-H(x, y)}{u-x}\leqq\frac{H(u, v)-H(x, v)}{u-x}$$

for $a\leqq x<u\leqq A$, $b\leqq y\leqq v\leqq B$. Taking the limit $u\to x$, we get

$$\partial_1 H(x, y)\leqq\partial_1 H(x, v).$$

Therefore the function $y\mapsto\partial_1 H(x,y)$, $b\leqq y\leqq B$ is not only continuous, but it is increasing for almost all $a\leqq x\leqq A$. Similarly, $x\mapsto\partial_2 H(x,y)$ is also increasing for $a\leqq x\leqq A$.

To prove (6), let φ be an arbitrary Young function. Then, by the Lemma, there exists an increasing sequence φ_n of elementary Young functions such that (3) holds. Thus, by the above properties of H,

$$\lim_{n\to\infty}\partial_1 H(t, \varphi_n(t))=\partial_1 H(t, \varphi(t)) \tag{7}$$

for almost all $a \leqq t \leqq A$ and

$$\lim_{n\to\infty} \partial_2 H(\varphi_n^{(-1)}(s), s) = \partial_2 H(\varphi^{(-1)}(s), s)$$

for almost all $b \leqq s \leqq B$. Since

$$\partial_1 H(t, b) \leqq \partial_1 H(t, \varphi_n(t)) \leqq \partial_1 H(t, \varphi(t)) \leqq \partial_1 H(t, B),$$

furthermore $\partial_1 H(t, b)$ and $\partial_1 H(t, B)$ are integrable functions on $[a, A]$, therefore the Lebesgue convergence theorem (see B. Sz.-Nagy [14]) can be applied. Thus, by (7), we get

$$\lim_{n\to\infty} \int_a^x \partial_1 H(t, \varphi_n(t))\, dt = \int_a^x \partial_1 H(t, \varphi(t))\, dt \tag{8}$$

for all $a \leqq x \leqq A$. Similarly,

$$\lim_{n\to\infty} \int_b^y \partial_2 H(\varphi_n^{(-1)}(s), s)\, ds = \int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds. \tag{9}$$

However, Theorem 1 yields

$$H(x, y) \leqq H(a, b) + \int_a^x \partial_1 H(t, \varphi_n(t))\, dt + \int_b^x \partial_2 H(\varphi_n^{(-1)}(s), s)\, ds$$

for all $n \in \mathbf{N}$, $a \leqq x \leqq A$, $b \leqq y \leqq B$. Letting $n \to \infty$ and using (8) and (9) we obtain (6), which was to be proved.

Remarks.

(i) Assuming only the existence of the partial derivatives $\partial_1 H(x, y)$, $\partial_2 H(x, y)$ of $H \in \mathscr{H}$, and using the same method one can prove that

$$H(x, y) \leqq H(a, b) + \int_a^x \partial_1 H(t, \varphi(t) - 0)\, dt + \int_b^x \partial_2 H(\varphi^{(-1)}(s) + 0, s)\, ds$$

holds for all Young functions φ.

(ii) The inequality (6) can be interpretated in the following way: The equation

$$m([x, u] \times [y, v]) = H(x, y) + H(u, v) - H(x, v) - H(u, y),$$
$$a \leqq x \leqq u \leqq A, \quad b \leqq y \leqq v \leqq B$$

defines a Lebesgue—Stieltjes measure on $[a, A] \times [b, B]$. If φ is a Young function then let

$$P_{x,y} := [a, x] \times [b, y],$$
$$Q_x := \{(s, t) | a \leqq s \leqq x,\ b \leqq t \leqq \varphi(s)\},$$
$$R_y := \{(s, t) | b \leqq t \leqq y,\ a \leqq s \leqq \varphi^{(-1)}(t)\}.$$

Now one can see that $P_{x,y} \subseteq Q_x \cup R_y$ for all $a \leqq x \leqq A$, $b \leqq y \leqq B$. Therefore

$$m(P_{x,y}) \leqq m(Q_x) + m(R_y). \tag{10}$$

Calculating the measures of these sets, one can show that (10) reduces to (6). Using the above argument, (6) was proved by the author [13] in the case when H is a C^2 function, i.e., when the measure m has the density function $\partial_1 \partial_2 H(x, y)$.

4. Young-type functional inequalities

Theorem 3. *Let H satisfy the conditions of Theorem 2; furthermore, let $f\colon [a, A] \to \mathbf{R}$ and $g\colon [b, B] \to \mathbf{R}$ be arbitrary functions. Then the functional inequality*

$$H(x, y) \leqq f(x) + g(y), \quad a \leqq x \leqq A, \quad b \leqq x \leqq B \tag{11}$$

is satisfied if and only if there exist two nonnegative functions $p\colon [a, A] \to [0, \infty[$, $q\colon [b, B] \to [0, \infty[$, a constant $\alpha \in \mathbf{R}$ and a Young function $\varphi\colon [a, A] \to [b, B]$ such that

$$f(x) = \int_a^x \partial_1 H(t, \varphi(t))\, dt + p(x) + \alpha, \quad a \leqq x \leqq A, \tag{12}$$

$$g(y) = \int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds + q(y) + H(a, b) - \alpha, \quad b \leqq y \leqq B. \tag{13}$$

Proof. The "if" part of the statement is a consequence of Theorem 2.

To prove the converse, assume that (11) is satisfied. Define $f_1\colon [a, A] \to \mathbf{R}$ by

$$f_1(x) := \sup_y (H(x, y) - g(y)). \tag{14}$$

Then (11) yields $f_1 \leqq f$. Therefore the function $p := f - f_1$ is nonnegative. Using the subadditivity of the sup operation and the estimate (7), we get

$$f_1(x) = \sup_y (H(x, y) - H(u, y) + H(u, y) - g(y)) \leqq \sup_y (H(x, y) - H(u, y)) + f_1(u) \leqq$$

$$\leqq \max\{|H(x, b) - H(u, b)|,\ |H(x, B) - H(u, B)|\} + f_1(u),$$

whence we obtain

$$|f_1(x) - f_1(u)| \leqq \max\{|H(x, b) - H(u, b)|,\ |H(x, B) - H(u, B)\} \tag{15}$$

for all $x, u \in [a, A]$. Since H is absolutely continuous on the boundary of $[a, A] \times \times [b, B]$ therefore (15) shows that f_1 is an absolutely continuous function. By (14) we have

$$H(x, y) \leqq f_1(x) + g(y) \quad a \leqq x \leqq A, \quad b \leqq x \leqq B.$$

Therefore the function $g_1\colon [b, B]\to\mathbf{R}$, defined by

$$g_1(y) := \sup_x \left(H(x, y) - f_1(x)\right), \tag{16}$$

satisfies $g_1 \leqq g$. Thus the function $q := g - g_1$ is nonnegative. A similar argument shows that g_1 is also an absolutely continuous function, and by (16) we have

$$H(x, y) \leqq f_1(x) + g_1(y), \quad a \leqq x \leqq A, \quad b \leqq y \leqq B. \tag{17}$$

Thus

$$f_1(x) = \sup_y \left(H(x, y) - g_1(y)\right) \geqq \sup_y \left(H(x, y) - g(y)\right) = f_1(x),$$

i.e.,

$$f_1(x) = \sup_y \left(H(x, y) - g_1(y)\right) \tag{18}$$

for all $a \leqq x \leqq A$. Write

$$\Phi := \{(x, y) | H(x, y) = f_1(x) + g_1(y)\}.$$

Since $x \mapsto H(x, y) - f_1(x)$ and $y \mapsto H(x, y) - g_1(y)$ are continuous functions, therefore the supremum in (16) and (18) is attained, i.e., for all x there exists y such that $(x, y) \in \Phi$, and for all y there exists x such that $(x, y) \in \Phi$.

The following estimate shows that H is a continuous function:

$$\begin{aligned} |H(x, y) - H(u, v)| &\leqq |H(x, y) - H(u, y)| + |H(u, y) - H(u, v)| \leqq \\ &\leqq \max\{|H(x, b) - H(u, b)|, \ |H(x, B) - H(u, B)|\} + \\ &+ \max\{|H(a, y) - H(a, v)|, \ |H(A, y) - H(A, v)|\}. \end{aligned}$$

Thus Φ is a closed set. Define $\varphi\colon [a, A]\to[b, B]$ by

$$\varphi(x) = \sup\{y | (x, y) \in \Phi\}.$$

Clearly, $(x, \varphi(x)) \in \Phi$, i.e.,

$$H(x, \varphi(x)) = f_1(x) + g_1(\varphi(x)), \quad a \leqq x \leqq A. \tag{19}$$

First we show that φ is a Young function. If φ were not increasing, then there would exist x, z such that $a \leqq x < z \leqq A$ and $\varphi(x) > \varphi(z)$. Then

$$\begin{aligned} -H(x, \varphi(x)) &= -f_1(x) - g_1(\varphi(x)), \\ H(x, \varphi(z)) &\leqq f_1(x) + g_1(\varphi(x)), \\ -H(z, \varphi(z)) &= -f_1(z) + g_1(\varphi(z)), \\ H(z, \varphi(x)) &< f_1(z) + g_1(\varphi(x)). \end{aligned}$$

Adding these inequalities, we get

$$H(x, \varphi(z)) + H(z, \varphi(x)) - H(x, \varphi(x)) - H(z, \varphi(z)) < 0,$$

which contradicts (4). Thus φ is an increasing function.

To prove the right continuity of φ, let x_0 be arbitrary and let x_n be a decreasing sequence with $\lim_{n\to\infty} x_n = x_0$. Then $\varphi(x_n)$ is convergent, write $y_0 := \lim_{n\to\infty} \varphi(x_n)$. Φ is closed and $(x_n, \varphi(x_n)) \in \Phi$ for all $n \in \mathbf{N}$, therefore $(x_0, y_0) \in \Phi$. Thus $\varphi(x_0) \geqq y_0$. On the other hand, $\varphi(x_0) \leqq \varphi(x_n)$ for all $n \in \mathbf{N}$, whence we get $\varphi(x_0) \leqq y_0$. So $\varphi(x_0) = y_0 = \lim_{n\to\infty} \varphi(x_n)$, which was to be proved.

Finally we show that $(a, b), (A, B) \in \Phi$. If $(a, b) \notin \Phi$, then

$$H(a, b) < f_1(a) + g_1(b).$$

However, by the properties of Φ, there exist $a \leqq x \leqq A$ and $b \leqq y \leqq B$, such that

$$-H(x, b) = -f_1(x) - g_1(b),$$
$$-H(a, y) = -f_1(a) - g_1(y),$$

and we also have (17). Adding these four inequalities, we obtain

$$H(a, b) + H(x, y) - H(x, b) - H(a, y) < 0,$$

which is a contradiction. Thus $(a, b) \in \Phi$. Similarly, one sees that $(A, B) \in \Phi$. This latter relation means that $\varphi(A) = B$. Thus we have proved that φ is a Young function.

Our next aim is to verify

$$H(\varphi^{-1}(y), y) = f_1(\varphi^{-1}(y)) + g_1(y), \quad b \leqq y \leqq B. \tag{20}$$

Assume the contrary, that for a value y

$$H(\varphi^{(-1)}(y), y) < f_1(\varphi^{(-1)}(y)) + g_1(y). \tag{21}$$

Write $x := \varphi^{-1}(y)$. Then, by (19), we have

$$-H(x, \varphi(x)) = -f_1(x) - g_1(\varphi(x)). \tag{22}$$

Now we distinguish two cases. If $x = \varphi^{(-1)}(y) = a$, then $y \leqq \varphi(a)$. By the properties of Φ, there exists a value $u > x = a$ such that

$$-H(u, y) = -f_1(u) - g_1(y) \tag{23}$$

and we also have

$$H(u, \varphi(a)) \leqq f_1(u) + g_1(\varphi(a)). \tag{24}$$

Adding (21), (22), (23) and (24) we get

$$H(a, y) + H(u, \varphi(a)) - H(a, \varphi(a)) - H(u, y) < 0,$$

which is a contradiction. Therefore (20) is valid if $\varphi^{(-1)}(y) = a$.

If $x = \varphi^{(-1)}(y) > a$, then, for $t < x$, the definition of $\varphi^{(-1)}(y)$ yields $\varphi(t) \leqq y$. This inequality must be strict. Indeed, if $\varphi(t_0) = y$ for some $a \leqq t_0 < x$, then $\varphi(t) = y$

for $t_0 \leqq t < x$, since φ is increasing. The points $(t, \varphi(t))$ are in Φ for $t_0 \leqq t < x$ thus, taking the limit $t \to t_0$, we find that $(x, y) \in \Phi$, i.e., $H(x, y) = f_1(x) + g_1(y)$. This contradicts (21), and proves

$$\varphi(t) < y \quad \text{for} \quad a \leqq t < x. \tag{25}$$

By the properties of Φ there exists a value $a \leqq u \leqq A$ such that $(u, y) \in \Phi$, i.e.,

$$-H(u, y) = -f_1(u) - g_1(y). \tag{26}$$

Then $\varphi(u) \geqq y$, thus (25) implies $x \leqq u$. Applying (17), we have

$$H(u, \varphi(x)) \leqq f_1(u) + g_1(\varphi(x)). \tag{27}$$

Adding the inequalities (21), (22), (26) and (27), we obtain

$$H(x, y) + H(u, \varphi(x)) - H(u, y) - H(x, \varphi(x)) < 0. \tag{28}$$

To get a contradiction we have only to show $\varphi(x) \geqq y$ (since then (28) cannot be valid). If $\varphi(x) = B$ then there is nothing to prove. If $\varphi(x) < B$, then $x < A$. Now $x < t \leqq A$ implies $y \leqq \varphi(t)$. Taking the limit $t \to x+0$ and using the right continuity of φ, we can see that $y \leqq \varphi(x)$. Thus the proof of (20) is complete.

Let $a < t < A$ be an arbitrary point where f_1 is differentiable. Then, by (19), the function

$$x \mapsto f_1(x) + g_1(\varphi(t)) - H(x, \varphi(t))$$

has a minimum at $x = t$. Therefore the derivative vanishes there:

$$f_1'(t) = \partial_1 H(t, \varphi(t)).$$

Since f_1 is absolutely continuous, we have

$$f_1(x) = \int_a^x f_1'(t)\, dt + f_1(a) = \int_a^x \partial_1 H(t, \varphi(t))\, dt + \alpha \tag{29}$$

for all $a \leqq x \leqq A$, where $\alpha = f_1(a)$. Similarly, it follows from (20) that

$$g_1(y) = \int_b^y \partial_2 H(\varphi^{(-1)}(s), s)\, ds + g_1(b). \tag{30}$$

However, as we have proved, $(a, b) \in \Phi$, that is

$$g_1(b) = H(a, b) - f_1(a) = H(a, b) - \alpha. \tag{31}$$

Since $f = f_1 + p$, $g = g_1 + q$, therefore (29), (30) and (31) show that (12) and (13) are satisfied.

The proof of the theorem is complete.

Remark. In the proof of the "only if" part of Theorem 3 we have not used all the regularity properties of H. We used only inequality (4) and that $\partial_1 H$ and $\partial_2 H$ exist everywhere.

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Best coapproximation and Schauder bases in Banach spaces

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1. Introduction

V. N. NIKOL'SKIĬ [8], [9] studied the problem of best approximation in Banach spaces with basis. The study was carried out further by J. R. RETHERFORD [13], [14]. He characterized (strictly) monotone bases and (strictly) comonotone bases by means of best approximation. He also characterized (strict) orthogonality and (strict) co-orthogonality in Banach spaces having unconditional bases by means of best approximation. Some further connections between best approximation and theory of bases can be found in the book of I. SINGER [17].

Another kind of approximation known as "Best coapproximation" was introduced by C. FRANCHETTI and M. FURI [2]. The work was continued on this topic by P. L. PAPINI and I. SINGER [10], [11], GEETHA S. RAO [3] and others. In this paper, some characterizations of bases in Banach spaces are obtained by means of best coapproximation. Certain kind of norms are introduced using best coapproximation in which the given bases are (strictly) monotone and (strictly) comonotone, respectively. Equivalent norms are provided in which the given bases possess the special properties. The analogous theory is detailed in Banach spaces having unconditional bases.

2. Notation and terminology

Let E be a Banach space. A sequence $\{x_n\}$ in E is a basis of E if for every $x \in E$ there exists a unique sequence of scalars $\{\alpha_n\} \subset \mathbf{K}$ such that

$$x = \sum_{i=1}^{\infty} \alpha_i x_i. \tag{1}$$

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A system (x_n, f_n), $\{x_n\} \subset E$, $\{f_n\} \subset E^*$ is biorthogonal if $f_i(x_j) = \delta_{ij}$. If (x_n, f_n) is a biorthogonal system with $\{x_n\}$ a basis in E, then (x_n, f_n) is a Schauder basis for E if for each $x \in E$,

(2) $$x = \sum_{i=1}^{\infty} f_i(x)\, x_i.$$

$\{f_n\} \subset E^*$ may some times be called an associated sequence of coefficient functionals (a.s.c.f.—here after). A sequence $\{x_n\}$ in E is a basic sequence if $\{x_n\}$ is a basis of the closed linear subspace $[x_n]$ of E where $[.]$ denotes the linear span of x_n's. A basis $\{x_n\}$ of E is unconditional if the convergence of (1) or (2) is unconditional, for each $x \in E$.

Let E be a Banach space, G be a linear subspace of E and $x \in E$. An element $g_0 \in G$ is a best approximation of x from G if

(3) $$\|x - g_0\| \leqq \|x - g\| \quad (g \in G).$$

The set of best approximations of x from G is denoted by $P_G(x)$. An element $g_0 \in G$ is a best coapproximation of x from G if

(4) $$\|g_0 - g\| \leqq \|x - g\| \quad (g \in G).$$

The set of best coapproximations of x from G is denoted by $R_G(x)$. For a sequence $\{x_n\}$ of E, let

$$G_n = [x_1, x_2, \ldots, x_n] \quad \text{and} \quad G^n = [x_{n+1}, x_{n+2}, \ldots].$$

Let $\mathscr{D} = \{\{i_1, i_2, \ldots, i_n\} \subset \mathscr{N} \mid 1 \leqq n < \infty\}$, where $\mathscr{N}$ denotes the set of all natural numbers. For a sequence $\{x_n\}$ of E, let

$$G_d = [x_i \colon i \in d] \quad \text{and} \quad G^d = [x_i \colon i \in \mathscr{N} \setminus d], \quad \text{for} \quad d \in \mathscr{D}.$$

A sequence $\{x_n\}$ is basic in E (an unconditional basic in E) if there is a K such that for every $m \leqq n$ ($d \subset d'$, $d' \in \mathscr{D}$) and arbitrary scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ ($\{\alpha_i\}_{i \in d'}$)

$$\Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\| \leqq K \Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| \quad \Big(\Big\|\sum_{i \in d} \alpha_i x_i\Big\| \leqq K \Big\|\sum_{i \in d'} \alpha_i x_i\Big\|\Big).$$

Let (x_n, f_n) be a Schauder basis for E. Then $s_n(x)$ and $r_n(x)$ (respectively $s_d(x)$, $r_d(x)$) are defined as

$$s_n(x) = \sum_{i=1}^{n} f_i(x) x_i \quad \text{and} \quad r_n(x) = x - s_n(x)$$

$$\Big(s_d(x) = \sum_{i \in d} f_i(x) x_i \quad \text{and} \quad r_d(x) = x - s_d(x)\Big).$$

3. Characterization of monotone bases

Definition 3.1. Let E be a Banach space with a basis $\{x_n\}$. Then E is said to satisfy *Property* (Λ_1) if there exists no collection of scalars $\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+m}$, for all $n, m \in \mathcal{N}$, such that $\sum_{i=n+1}^{n+m} |\alpha_i| \neq 0$ and satisfying

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| = \left\|\sum_{i=1}^{n+m} \alpha_i x_i\right\| = \left\|\sum_{i=1}^{n} \alpha_i x_i + \frac{1}{2} \sum_{i=m+1}^{n+m} \alpha_i x_i\right\|.$$

Definition 3.2. Let E be a Banach space with a basis $\{x_n\}$. Then E is said to satisfy *Property* (Λ_2) if there exists no collection of scalars $\alpha_{m+1}, \alpha_{m+2}, \ldots, a_n$, for all $n, m \in \mathcal{N}$, such that $\sum_{i=m+1}^{n} |\alpha_i| \neq 0$ and satisfying

$$\left\|\sum_{i=n+1}^{\infty} \alpha_i x_i\right\| = \left\|\sum_{i=m+1}^{\infty} \alpha_i x_i\right\| = \left\|\sum_{i=n+1}^{\infty} \alpha_i x_i + \frac{1}{2} \sum_{i=m+1}^{n} \alpha_i x_i\right\|.$$

Remark 3.1. All Banach spaces having basis $\{x_n\}$ and are strictly convex satisfy Property (Λ_1) and Property (Λ_2). But there is no connection between Property (Λ_1) and Property (Λ_2) in general.

Following [15], the next definition is introduced.

Definition 3.3. If $\{x_n\}$ is a basis for a Banach space E, then

(i) $\{x_n\}$ is *monotone* if $\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \leqq \left\|\sum_{i=1}^{n+m} \alpha_i x_i\right\|$ for all n and for all collections of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m} \in \mathbf{K}$.

(ii) $\{x_n\}$ is *strictly monotone* if strict inequality holds in (i) whenever $\sum_{i=n+1}^{n+m} |\alpha_i| \neq 0$.

(iii) $\{x_n\}$ is *comonotone* if $\left\|\sum_{i=n}^{\infty} \alpha_i x_i\right\| \leqq \left\|\sum_{i=m}^{\infty} \alpha_i x_i\right\|$ whenever $\sum_{i=n}^{\infty} \alpha_i x_i$ converges and for all collections of scalars $\alpha_m, \alpha_{m+1}, \ldots, \alpha_n, \alpha_{n+1} \ldots \in \mathbf{K}$.

(iv) $\{x_n\}$ is *strictly comonotone* if strict inequality holds in (iii) whenever $\sum_{i=m}^{n-1} |\alpha_i| \neq 0$.

Theorem 3.1. *Let E be a Banach space with a basis $\{x_n\}$. The following statements are true about $\{x_n\}$:*

(i) *$\{x_n\}$ is monotone if and only if $R_{G_n}(x) = \{s_n(x)\}$ $n = 1, 2, \ldots$.*

(ii) *$\{x_n\}$ is strictly monotone if and only if $R_{G_n}(x) = \{s_n(x)\}$ $n = 1, 2, \ldots$ and E satisfies Property (Λ_1).*

(iii) *$\{x_n\}$ is comonotone if and only if $R_{G^n}(x) = \{r_n(x)\}$ $n = 1, 2, \ldots$.*

(iv) *$\{x_n\}$ is strictly comonotone if and only if $R_{G^n}(x) = \{r_n(x)\}$ $n = 1, 2, \ldots$ and E satisfies Property (Λ_2).*

Proof. (i) $\{x_n\}$ is monotone, then $\|\sum_{i=1}^{n}\alpha_i x_i\| \leqq \|\sum_{i=1}^{n+1}\alpha_i x_i\|$ for all scalars $\alpha_{n+1}\in\mathbf{K}$. Then, if $x=\sum_{i=1}^{\infty}\alpha_i x_i\in E$, it follows that

$$\|\sum_{i=1}^{n}\alpha_i x_i-\sum_{i=1}^{n}\beta_i x_i\| = \|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i\| \leqq \|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i+\alpha_{n+1}x_{n+1}\| \leqq \dots$$

$$\dots \leqq \|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i+\sum_{i=n+1}^{\infty}\alpha_i x_i\| = \|x-\sum_{i=1}^{n}\beta_i x_i\|$$

for all $\sum_{i=1}^{n}\beta_i x_i=p(\neq s_n(x))\in G_n$. Thus $s_n(x)\in R_{G_n}(x)$. On the other hand, $E=G_n\oplus \oplus[x_{n+1}, x_{n+2}, \dots]$, $R_{G_n}^{-1}(0)\supset[x_{n+1},\dots]$ and $R_{G_n}^{-1}(0)\cap G_n=\{0\}$, where $R_{G_n}^{-1}(0)= =\{y\in E \mid R_{G_n}(y)\ni 0\}$. Therefore $E=G_n\oplus R_G^{-1}(0)$ and $R_{G_n}(x)$ is unique for every $x\in E\setminus G_n$ (from [3]), i.e. $R_{G_n}(x)=\{s_n(x)\}$ as $s_n(x)\in R_{G_n}(x)$ $(x\in E)$.

If $R_{G_n}(x)=\{s_n(x)\}$, then for $x=\sum_{i=1}^{n+m}\alpha_i x_i$, it follows that

$$\|\sum_{i=1}^{n}\alpha_i x_i-p\| \leqq \|\sum_{i=1}^{n+m}\alpha_i x_i-p\| \quad (p\in G_n).$$

Since $0\in G_n$, it follows that

$$\|\sum_{i=1}^{n}\alpha_i x_i\| \leqq \|\sum_{i=1}^{n+m}\alpha_i x_i\|$$

for all collections of scalars $\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m}\in\mathbf{K}$. Thus $\{x_n\}$ is monotone.

(ii) If $R_{G_n}(x)=\{s_n(x)\}$, then

$$\|\sum_{i=1}^{n}\alpha_i x_i\| \leqq \|\sum_{i=1}^{n+m}\alpha_i x_i\|$$

for all collections of scalars $\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m}\in\mathbf{K}$ was proved.

If equality holds for some collection of scalars (say) $\alpha_1, \alpha_2, \dots, \alpha_n$, $\alpha_{n+1}, \dots, \alpha_{n+m}\in\mathbf{K}$, with $\sum_{i=n+1}^{n+m}|\alpha_i|\neq 0$, then

$$\|\sum_{i=1}^{n}\alpha_i x_i\| = \|\sum_{i=1}^{n+m}\alpha_i x_i\|.$$

Consider $\left\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=1}^{n+m}\alpha_i x_i\right\|$. Since

$$R_{G_n}\left(\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i x_i, \tag{5}$$

it follows that

$$\Big\|\sum_{i=1}^{n}\alpha_i x_i\Big\| \leqq \Big\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\Big\|.$$

On the other hand,

$$\Big\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\Big\| \leqq \Big\|\frac{1}{2}\sum_{i=1}^{n}\alpha_i x_i\Big\|+\Big\|\frac{1}{2}\sum_{i=1}^{n+m}\alpha_i x_i\Big\| = \Big\|\sum_{i=1}^{n}\alpha_i x_i\Big\|. \tag{6}$$

From (5) and (6), it follows that

$$\Big\|\sum_{i=1}^{n}\alpha_i x_i\Big\| = \Big\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\Big\| = \Big\|\sum_{i=1}^{n+m}\alpha_i x_i\Big\|$$

for some collection of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m} \in \mathbf{K}$ with $\sum_{i=n+1}^{n+m}|\alpha_i| \neq 0$, in contradiction to the Property (Λ_1) satisfied by E. Thus $\{x_n\}$ is strictly monotone.

Proceeding to the other implication, if

$$\Big\|\sum_{i=1}^{n}\alpha_i x_i\Big\| < \Big\|\sum_{i=1}^{n+m}\alpha_i x_i\Big\|$$

for all collections of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m} \in \mathbf{K}$ with $\sum_{i=n+1}^{n+m}|\alpha_i| \neq 0$, then it is clear that it implies $R_{G_n}(x)=\{s_n(x)\}$ and Property (Λ_1) for E.

(iii) Consider $\{x_n\}$ is comonotone. Then

$$\Big\|\sum_{i=n}^{\infty}\alpha_i x_i\Big\| \leqq \Big\|\sum_{i=m}^{\infty}\alpha_i x_i\Big\|$$

for all collections of scalars $\alpha_m, \alpha_{m+1}, \ldots, \alpha_n, \ldots \in \mathbf{K}$ for which $\sum_{i=n}^{\infty}\alpha_i x_i$ is convergent. Then for $x=\sum_{i=1}^{\infty}\alpha_i x_i \in E$, it follows that,

$$\Big\|r_n(x)-\sum_{i=n+1}^{\infty}\beta_i x_i\Big\| = \Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i-\sum_{i=n+1}^{\infty}\beta_i x_i\Big\| \leqq \Big\|\sum_{i=n+1}^{\infty}(\alpha_i-\beta_i)x_i+\alpha_n x_n\Big\| \leqq \ldots$$

$$\ldots \leqq \Big\|\sum_{i=n+1}^{\infty}(\alpha_i-\beta_i)x_i+\sum_{i=1}^{n}\alpha_i x_i\Big\| = \Big\|x-\sum_{i=n+1}^{\infty}\beta_i x_i\Big\|$$

for all $\sum_{i=n+1}^{\infty}\beta_i x_i = p(\neq r_n(x)) \in G^n$. Thus $r_n(x) \in R_{G^n}(x)$. But $E=G^n \oplus [x_1, x_2, \ldots, x_n]$, $R_{G^n}^{-1}(0) \supset [x_1, x_2, \ldots, x_n]$ and $R_{G^n}^{-1}(0) \cap G^n = \{0\}$. Therefore $E=G^n \oplus R_{G^n}^{-1}(0)$ and $R_{G^n}(x)$ is unique for all $x \in E \setminus G^n$. Thus $R_{G^n}(x)=\{r_n(x)\}$.

On the other hand if $R_{G^n}(x)=\{r_n(x)\}$, then for $x=\sum_{i=m+1}^{\infty}\alpha_i x_i$, it follows that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i-p\Big\|\leqq\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i-p\Big\|$$

for all $p\in G^n$. Since $0\in G^n$, it follows that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i\Big\|$$

for all collections of scalars $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ for which $\sum_{i=n+1}^{\infty}\alpha_i x_i$ is convergent.

(iv) If $R_{G^n}(x)=\{r_n(x)\}$, then

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i\Big\|$$

for all scalars $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ for which $\sum_{i=n+1}^{\infty}\alpha_i x_i$ is convergent. If equality holds for some $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ (say) with $\sum_{i=m+1}^{n}|\alpha_i|\neq 0$, then

$$\Big\|\sum_{i=n+1}^{\infty}a_i x_i\Big\|=\Big\|\sum_{i=m+1}^{\infty}a_i x_i\Big\|.$$

Consider $\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i$. Since

$$R_{G^n}\Big(\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big)=\sum_{i=n+1}^{\infty}\alpha_i x_i,$$

it is clear that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big\|. \tag{7}$$

On the other hand,

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big\|\leqq\frac{1}{2}\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|+\frac{1}{2}\Big\|\sum_{i=m+1}^{n}\alpha_i x_i\Big\|=\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|. \tag{8}$$

From (7) and (8), it follows that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|=\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big\|=\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i\Big\|$$

for some collection of scalars $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ with $\sum_{i=m+1}^{n}|\alpha_i|\neq 0$, contradicting Property (Λ_2) satisfied by E. Hence $\{x_n\}$ is strictly comonotone.

Proceeding to the other implication, if $\left\|\sum_{i=n+1}^{\infty} \alpha_i x_i\right\| < \left\|\sum_{i=m+1}^{\infty} \alpha_i x_i\right\|$ for all collections of scalars $\alpha_{m+1}, \ldots, \alpha_{n+1}, \ldots \in \mathbf{K}$ with $\sum_{i=m+1}^{n} |\alpha_i| \neq 0$ and $\sum_{i=n+1}^{\infty} \alpha_i x_i$ is convergent, then it is clear that $R_{G^n}(x) = \{r_n(x)\}$ for $x \in E$ and E satisfies Property (Λ_2).

Definition 3.4. The norm in a Banach space E with a basis $\{x_n\}$ is called a *CT-norm* (with respect to the basis $\{x_n\}$) if

(a) for every $x \in E$ and $n = 1, 2, \ldots$, there exists a unique polynomial $R_{G_n}(x) = \{s_n(x)\}$ of best coapproximation to x.

(b) E satisfies Property (Λ_1).

Observe that CT-norms will be denoted by $\|\cdot\|_{\mathrm{CT}}$.

Definition 3.5. The norm in a Banach space E with a basis $\{x_n\}$ is called a *CK-norm* (with respect to the basis $\{x_n\}$) if

(a) for every $x \in E$, and $n = 1, 2, \ldots$ there exists a unique polynomial complement $R_{G^n}(x) = \{r_n(x)\}$ of best coapproximation to x.

(b) E satisfies Property (Λ_2).

Note that CK-norms are denoted by $\|\cdot\|_{\mathrm{CK}}$.

Definition 3.6. The norm in a Banach space E with a basis $\{x_n\}$ is called a *CTK-norm* (with respect to the basis $\{x_n\}$) if it is simultaneously a CT-norm and a CK-norm with respect to this basis.

CTK-norms are denoted by $\|\cdot\|_{\mathrm{CTK}}$.

Lemma 3.1. *Let E be a Banach space with a basis $\{x_n\}$. The following statements are true:*

(i) *The norm in E is a CT-norm if and only if*

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| < \left\|\sum_{i=1}^{n+1} \alpha_i x_i\right\|$$

for all collections of scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbf{K}$ with $\alpha_{n+1} \neq 0$.

(ii) *The norm in E is a CK-norm if and only if*

$$\left\|\sum_{i=n+1}^{\infty} \alpha_i x_i\right\| < \left\|\sum_{i=n}^{\infty} \alpha_i x_i\right\|$$

for all sequence of scalars $\{\alpha_n\}_n^{\infty}$ with $\alpha_n \neq 0$ for which the series $\sum_{i=n}^{\infty} \alpha_i x_i$ is convergent.

(iii) *If the norm in E is CTK-norm, then*

$$\Big\|\sum_{i=l+1}^{n} \alpha_i x_i\Big\| < \Big\|\sum_{i=l}^{n+1} \alpha_i x_i\Big\|$$

for all collections of scalars $\alpha_l, \alpha_{l+1}, \dots, \alpha_n, \alpha_{n+1} \in \mathrm{K}$ *with* $|\alpha_l|+|\alpha_{n+1}|\neq 0$.

Proof. The proof is clear from the proof of Theorem 3.1.

Example 3.1. A CK-norm which is not a CT-norm: The numbers

$$\|x\|_{\mathrm{CK}} = \max_{1\leqq n<\infty}\left(\frac{1}{n}\sum_{i=1}^{n}|y_i| + \sup_{n+1\leqq j<\infty}|y_j|\right) \quad (x=(y_i)\in c_0)$$

define a norm on c_0, equivalent to the initial norm of c_0. This norm $\|\cdot\|_{\mathrm{CK}}$ is a CK-norm but not a CT-norm with respect to the unit vector basis $\{x_n\}$ of c_0. On the other hand, it follows that

$$\|x_1+x_2\|_{\mathrm{CK}} = \max\left(1+1, \frac{1}{2}(1+1), \frac{1}{3}(1+1), \dots\right) = 2$$

$$\|x_1+x_2+x_3\|_{\mathrm{CK}} = \max\left(1+1, \frac{1}{2}(1+1)+1, \frac{1}{3}(1+1+1), \frac{1}{4}(1+1+1), \dots\right) = 2.$$

Hence by Lemma 3.1, this is not a CT-norm.

Example 3.2. A CT-norm which is not a CK-norm: For every integer $n\geqq 2$, let $\pi_{1,n}$ denote the collection of all permutations of the set

$$\{2, 3, \dots, n-1, n+1, n+2, \dots\}.$$

Then the numbers

$$\|x\|_{\mathrm{CT}} = \sup_{2\leqq n<\infty}\ \sup_{d\in\pi_{1,n}}\left(\frac{|y_1|}{n2^n} + \sum_{i=2}^{\infty}\frac{|y_{d(i)}|}{2^i}\right) \quad (x=(y_i)\in c_0)$$

define a norm on c_0, equivalent to the initial norm of c_0. This is a CT-norm but not a CK-norm with respect to the unit vector basis $\{x_n\}$ of c_0. The violation in the characterizing inequality of CK-norm in Lemma 3.1 was shown by I. SINGER [17].

Remark 3.2. The above examples show that there is no relation between CT-norms and CK-norms. That there can exist a basis and a norm which is a CT-norm but not a CK-norm and vice versa can be observed by the above examples.

Theorem 3.2. *Let E be a Banach space with a basis* $\{x_n\}$ *and let* $\{f_n\}\subset E^*$ *be the a.s.c.f. Then the following statements hold.*

(i) *A CT-norm on E equivalent to the initial norm on E can be introduced by the formula*

$$\|x\|_{\mathrm{CT}} = \sum_{i=1}^{\infty}\frac{1}{2^i}\|f_i(x)x_i\| + \sup_{1\leqq n<\infty}\Big\|\sum_{i=1}^{n} f_i(x)\,x_i\Big\| \tag{9}$$

(ii) *A CK-norm on E equivalent to the initial norm on E can be introduced by the formula*

$$\|x\|_{\mathrm{CK}} = \max_{1 \leqq n < \infty} \left(\frac{1}{n} \sum_{i=1}^{n} \|f_i(x) x_i\| + \Big\| \sum_{i=n+1}^{\infty} f_i(x) x_i \Big\| \right) \tag{10}$$

and also another equivalent CK-norm on E, by the formula

$$\|x\|_{\mathrm{CK}} = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x) x_i\| + \sup_{1 \leqq n < \infty} \Big\| \sum_{i=1}^{\infty} f_i(x) x_i \Big\| \tag{11}$$

(iii) *A CTK-norm on E equivalent to initial norm on E can be introduced by the formula*

$$\|x\|_{\mathrm{CTK}} = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x) x_i\| + \sup_{1 \leqq n, m < \infty} \Big\| \sum_{i=1}^{n} f_i(x) x_i \Big\| \tag{12}$$

and another equivalent CTK-norm on E by the formula

$$\|x\|_{\mathrm{CTK}} = \sup_{1 \leqq n < \infty} \Big\{ \Big\| \sum_{i=1}^{n} f_i(x) x_i \Big\|_{\mathrm{CK}} + \Big\| \sum_{i=n+1}^{\infty} f_i(x) x_i \Big\|_{\mathrm{CT}} \Big\}. \tag{13}$$

Proof. The fact that all these numbers define a norm and all these norms are equivalent to the original norm on E was proved previously and can be found in [17]. Now that they actually have the property of CT, CK, CTK-norms is proved here.

(i)
$$\Big\| \sum_{i=1}^{n} \alpha_i x_i \Big\|_{\mathrm{CT}} = \sum_{i=1}^{n} \frac{1}{2^i} \|\alpha_i x_i\| + \max_{1 \leqq k \leqq n} \Big\| \sum_{i=1}^{k} \alpha_i x_i \Big\| <$$
$$< \sum_{i=1}^{n+1} \frac{1}{2^i} \|\alpha_i x_i\| + \max_{1 \leqq k \leqq n+1} \Big\| \sum_{i=1}^{k} \alpha_i x_i \Big\| = \Big\| \sum_{i=1}^{n+1} \alpha_i x_i \Big\|_{\mathrm{CT}}$$

for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} \in \mathbf{K}$ with $\alpha_{n+1} \neq 0$. Hence by Lemma 3.1, it follows that it is a CT-norm.

(ii). (ii)$_1$. Let $\{a_i\}_{i-1}^{\infty}$ be a sequence of scalars with $\alpha_{l-1} \neq 0$, such that $\sum_{i=l}^{\infty} \alpha_i x_i$ converges. Then it follows that for a suitable number n_0 with $l \leqq n_0 < \infty$,

$$\Big\| \sum_{i=l}^{\infty} \alpha_i x_i \Big\|_{\mathrm{CK}} = \frac{1}{n_0} \sum_{i=l}^{n_0} \|\alpha_i x_i\| + \Big\| \sum_{i=n_0+1}^{\infty} \alpha_i x_i \Big\| <$$
$$< \frac{1}{n_0} \sum_{i=l-1}^{n_0} \|\alpha_i x_i\| + \Big\| \sum_{i=n_0+1}^{\infty} \alpha_i x_i \Big\| \leqq \max_{l-1 \leqq n < \infty} \left(\frac{1}{n} \sum_{i=l-1}^{n} \|\alpha_i x_i\| + \Big\| \sum_{i=n+1}^{\infty} \alpha_i x_i \Big\| \right) =$$
$$= \Big\| \sum_{i=l-1}^{\infty} \alpha_i x_i \Big\|_{\mathrm{CK}}.$$

Hence from Lemma 3.1, it follows that this norm is a CK-norm.

$(\mathrm{ii})_2$.
$$\Big\|\sum_{i=l}^{\infty}\alpha_i x_i\Big\|_{\mathrm{CK}} = \sum_{i=l}^{\infty}\frac{1}{2^i}\|\alpha_i x_i\| + \max_{l\leqq n<\infty}\Big\|\sum_{i=n}^{\infty}\alpha_i x_i\Big\| <$$
$$< \sum_{i=l-1}^{\infty}\frac{1}{2^i}\|\alpha_i x_i\| + \max_{l-1\leqq n<\infty}\Big\|\sum_{i=n}^{\infty}\alpha_i x_i\Big\| = \Big\|\sum_{i=l-1}^{\infty}\alpha_i x_i\Big\|_{\mathrm{CK}}.$$

Hence from Lemma 3.1, it is clear that this norm is a CK-norm.

4. Characterization of bases

For a sequence $\{y_n\}$ in a Banach space E, let $P_n=[y_i\colon i\leqq n]$ and let $P=\bigcup_{n=1}^{\infty}P_n$. For $p\in P$, $p=\sum_{i=1}^{n}\alpha_i y_i$ for some n, let

$$s_m^n(p)=\begin{cases}\sum_{i=1}^{m}\alpha_i y_i & \text{if } m<n\\ p & \text{if } m\geqq n\end{cases}$$

and

$$r_m^n(p)=p-s_m^n(p).$$

Definition 4.1. The norm $\|\cdot\|$ of E is a

(i) *weak CT-norm* relative to $\{y_n\}$ if for each polynomial $p\in P$, $p=\sum_{i=1}^{n}\alpha_i y_i$ and each $m\leqq n$, the polynomial $\sum_{i=1}^{m}\alpha_i y_i$ is the unique best coapproximation to p from $[y_i\colon i\leqq m]$.

(ii) *weak CK-norm* relative to $\{y_n\}$ if for each polynomial $p\in P$, $p=\sum_{i=1}^{n}\alpha_i y_i$ and each $m\leqq n$ the complementary polynomial $\sum_{i=m+1}^{n}\alpha_i y_1$ is the best coapproximation to p from $[y_i\colon m+1\leqq i\leqq n]$.

(iii) *weak CTK-norm* relative to $\{y_n\}$ if it is simultaneously a weak CK-norm and a weak CT-norm relative to $\{y_n\}$.

Remark 4.1. It is clear that if $\{x_n\}$ is a basis for E, then a CT-, CK-, CTK-norm with respect to $\{x_n\}$ is a weak CT-norm, weak CK-norm and weak CTK-norm relative to $\{x_n\}$. Example 3.1 shows that the converse is false.

Theorem 4.1. *Let $\{y_n\}$ be a non-zero sequence in a Banach space E with the norm $\|\cdot\|$. Then the following statements hold.*

(i) *The norm is a weak CT-norm relative to* $\{y_n\}$ *if and only if*

$$(*_1) \qquad \sup_n \sup \{\|s_n(p)\colon p\in P,\ \|p\| \leqq 1\} = 1.$$

(ii) *The norm is a weak CK-norm relative to* $\{y_n\}$ *if and only if*

$$(*_2) \qquad \sup_n \sup \{\|r_n(p)\|\colon p\in P,\ \|p\| \leqq 1\} = 1.$$

(iii) *The norm is a weak CTK-norm relative to* $\{y_n\}$ *if and only if*

$$\max [(*_1), (*_2)] = 1.$$

Remark 4.2. Here $s_n(p)$ and $r_n(p)$ will assume the roles of $s_n^k(p)$ and $r_n^k(p)$ whenever $p\in P$ is expressible in the form $p=\sum_{i=1}^{k} \alpha_i y_i$ for some $k\in\mathcal{N}$ and $(*_1)$ and $(*_2)$ denote the expressions on the left-hand side of the equations.

Proof of Theorem 4.1. (1) Suppose that $p=\sum_{i=1}^{n} \alpha_i y_i\in P$ and $(*_1)=1$. Let $\gamma=\sum_{i=1}^{m} \beta_i y_i\in P_m$. If $\|p-\gamma\|\neq 0$, let $p'=\|p-\gamma\|^{-1}(p-\gamma)$. Then it follows that $\|p'\|=1$. Therefore, $\|s_m^n(p')\|\leqq 1$ by property $(*_1)=1$. But $\left\|s_m^n(\|p-\gamma\|^{-1}(p-\gamma))\right\|\leqq 1$ implies

$$\|s_m^n(p-\gamma)\| \leqq \|p-\gamma\|,$$

i.e.

$$\|s_m^n(p)-\gamma\| \leqq \|p-\gamma\|,$$

i.e. $s_m^n(p)$ is a best coapproximation to p. But since $P_n=P_m\oplus[x_{m+1}, \ldots, x_n]$, it follows that $R_{P_m}(p)$ is unique. Therefore $s_m^n(p)$ is the unique best coapproximation to p. On the other hand, if $\|p-\gamma\|=0$, then $s_m^n(p)=p=\gamma$ and the result is trivial.

Conversely if $s_m^n(p)$ is the unique best coapproximation to $p=\sum_{i=1}^{n} \alpha_i y_i$ for $m\leqq n$ and for $\|p\|\leqq 1$, it follows that $\|s_m^n(p)\|\leqq\|p\|\leqq 1$. Since only finite sums are dealt with, a $p\in P$ and n can be found such that $\|s_n(p)\|$ is nearly 1. Thus $(*_1)=1$.

(ii) and (iii). The proofs of (ii) and (iii) are similar and are omitted.

Theorem 4.2. *Let E be a Banach space with a basis $\{x_n\}$. A norm on E is a weak CK-norm if and only if*

$$\Big\|\sum_{i=m}^{n} \alpha_i y_i\Big\| \leqq \Big\|\sum_{i=m-1}^{n} \alpha_i y_i\Big\|$$

for arbitrary scalars $\alpha_{m-1}, \alpha_m, \ldots, \alpha_n, \alpha_{m-1}\neq 0$ *and* $m, n=1, 2, \ldots$.

Proof. The proof is similar to that of (ii) of Lemma 3.1.

Theorem 4.3. *The following statements about* $\{x_n\}$ *a sequence in a Banach space* E *with* $[x_i, i\in\mathcal{N}]=E$ *are equivalent:*

(i) $\{x_n\}$ *is a basis for* E.

(ii) *A weak CT-norm can be introduced relative to* $\{x_n\}$ *on* E *equivalent to the original norm on* E.

(iii) *A weak CK-norm can be introduced relative to* $\{x_n\}$ *on* E *equivalent to the original norm on* E.

(iv) *A weak CTK-norm can be introduced relative to* $\{x_n\}$ *on* E *equivalent to the original norm on* E.

Proof. (i) implies the other three was proved in the stronger form in Section 3 of this paper. If (ii) and (iii) implies (i), then (iv) also implies (i). So (ii) implies (i) is proved here as the other implication is similar.

Suppose $p\leqq q$, $\sum_{i=1}^{q}\alpha_i x_i\neq 0$, then by Theorem 4.1, it follows that

$$\Big\|s_p^q\Big(\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|^{-1}\sum_{i=1}^{q}\alpha_i x_i\Big)\Big\|\leqq 1$$

(i.e.)

$$\Big\|s_p^q\Big(\sum_{i=1}^{q}\alpha_i x_i\Big)\Big\|\leqq\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|$$

(i.e.)

$$\Big\|\sum_{i=1}^{p}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|.$$

If $\sum_{i=1}^{q}\alpha_i x_i=0$, then, since the norm is a weak CT-norm,

$$\Big\|\sum_{i=1}^{p}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|=0$$

implying

$$\sum_{i=1}^{p}\alpha_i x_i=0$$

for all $p\leqq q$. Thus Grinblyum's K-condition is satisfied with $K=1$.

5. Characterising orthogonal bases

Definition 5.1. Let E be a Banach space having a sequence $\{x_n\}$. Following [17],

(i) $\{x_n\}$ is *orthogonal* provided $\Big\|\sum_{i\in d_1}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i\in d_2}\alpha_i x_i\Big\|$ for arbitrary $d_1, d_2\in\mathcal{D}$, with $d_1\subset d_2$ and arbitrary collection of scalars $\{\alpha_i\}_{i\in d_2}$.

(ii) $\{x_n\}$ is *strictly orthogonal* if the inequality is strict whenever $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$.

(iii) $\{x_n\}$ is *coorthogonal* if $\left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i\right\|\leqq\left\|\sum_{i\in\mathcal{N}\setminus d_1}\alpha_i x_i\right\|$ for arbitrary $d_1, d_2\in\mathcal{D}$ with $d_1\subset d_2$ and arbitrary collection of scalars $\{\alpha_i\}$ for which $\sum_{i\in\mathcal{N}}\alpha_i x_i$ is convergent.

(iv) $\{x_n\}$ is strictly coorthogonal if the inequality of (iii) is strict whenever $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$.

Theorem 5.1. *Let E be a Banach space having an unconditional basis $\{x_n\}$. Then the following statements are true:*

(i) *$\{x_n\}$ is orthogonal if and only if $R_{G_d}(x)=\{s_d(x)\}$ for all $d\in\mathcal{D}$.*

(ii) *$\{x_n\}$ is strictly orthogonal if and only if $R_{G_d}(x)=\{s_d(x)\}$ for all $d\in\mathcal{D}$ and E has the property that there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, for all $d_2, d_1\in\mathcal{D}$ with $d_2\supset d_1$ and $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$ satisfying*

$$\left\|\sum_{i\in d_1}\alpha_i x_i\right\| = \left\|\sum_{i\in d_2}\alpha_i x_i\right\| = \left\|\sum_{i\in d_1}\alpha_i x_i+\frac{1}{2}\sum_{i\in d_2\setminus d_1}\alpha_i x_i\right\|.$$

(iii) *$\{x_n\}$ is strictly coorthogonal if and only if $R_{G^d}(x)=\{r_d(x)\}$ for all $d\in\mathcal{D}$ and E has the property that there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, for all $d_2, d_1\in\mathcal{D}$ with $d_2\supset d_1$ and $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$ satisfying*

$$\left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_1}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i+\frac{1}{2}\sum_{i\in d_2\setminus d_1}\alpha_i x_i\right\|.$$

Proof. The proof is similar to Theorem 3.1 and is omitted.

Remark 5.1. Since the notions of orthogonal and coorthogonal bases are equivalent, the characterization of coorthogonal bases is omitted in Theorem 5.1. Analogous to Definitions 3.4, 3.5 and 3.6, one may call the norms satisfying the "if" parts of (ii), (iii) and (ii) and (iii) of Theorem 5.1 as CNT-, CNK-, CNTK-norms respectively. While every CNK-norm is a CNT-norm, the converse is not always true. Example 3.1 illustrates this.

Theorem 5.2. *Let E be a Banach space with an unconditional basis $\{x_n\}$. Then every norm in E in which $R_{G^d}(x)=\{r_d(x)\}$ and there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, for all $d_2, d_1\in\mathcal{D}$ with $d_2\supset d_1$ and $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$ satisfying*

$$\left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_1}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i+\frac{1}{2}\sum_{i\in d_2\setminus d_1}\alpha_i x_i\right\|$$

also has $R_{G_d}(x)=\{s_d(x)\}$ and the property that there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, *for all* $d_1, d_2 \in \mathscr{D}$ *with* $d_2 \supset d_1$ *and* $\sum_{i \in d_2 \setminus d_1} |\alpha_i| \neq 0$ *satisfying*

$$\Big\|\sum_{i\in d_1} \alpha_i x_i\Big\| = \Big\|\sum_{i\in d_2} \alpha_i x_i\Big\| = \Big\|\sum_{i\in d_1} \alpha_i x_i + \frac{1}{2} \sum_{i\in d_2\setminus d_1} \alpha_i x_i\Big\|.$$

Proof. Every strictly coorthogonal basis is strictly orthogonal. This was proved by RETHERFORD [14]. Hence the theorem follows.

Theorem 5.3. *Let E be a Banach space with an unconditional basis* $\{x_n\}$ *and let* $\{f_n\} \subset E^*$ *be the a.s.c.f. Then a norm* $\|\cdot\|_*$ *on E can be introduced, equivalent to the initial norm on E, in which* $R_{G^d}(x) = \{r_d(x)\}$ *and there exist no scalars* $\{\alpha_i\}_{i \in d_2 \setminus d_1}$, *for all* $d_2, d_1 \in \mathscr{D}$ *with* $d_2 \supset d_1$ *and* $\sum_{i \in d_2 \setminus d_1} |\alpha_i| \neq 0$ *satisfying*

$$\Big\|\sum_{i\in \mathscr{N}\setminus d_2} \alpha_i x_i\Big\| = \Big\|\sum_{i\in \mathscr{N}\setminus d_1} \alpha_i x_i\Big\| = \Big\|\sum_{i\in \mathscr{N}\setminus d_2} \alpha_i x_i + \frac{1}{2} \sum_{i\in d_2\setminus d_1} \alpha_i x_i\Big\|,$$

by the formula

$$\|x\|_* = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x) x_i\| + \sup_{\{i_1, i_2, \dots, i_n\} \in \mathscr{D}} \Big\|\sum_{j=1}^{n} f_{i_j}(x) x_{i_j}\Big\|.$$

Proof. The equivalence of norms follows from I. SINGER [17, p. 554]. To prove that $\|\cdot\|_*$ has the required properties, it will be sufficient by Theorem 5.1 to prove that $\{x_n\}$ is strictly coorthogonal in this norm. Let $d_1, d_2 \in \mathscr{D}$ with $d_1 \subset d_2$ and $\sum_{i \in d_2 \setminus d_1} |\alpha_i| \neq 0$ be such that $\sum_{i \in \mathscr{N} \setminus d_2} \alpha_i x_i$ converges. Then it follows that

$$\mathscr{N} \setminus d_1 = (\mathscr{N} \setminus d_2) \cup (d_2 \setminus d_1).$$

Hence

$$\Big\|\sum_{i\in \mathscr{N}\setminus d_2} \alpha_i x_i\Big\|_* = \sum_{i\in \mathscr{N}\setminus d_2} \frac{1}{2^i} \|\alpha_i x_i\| + \sup_{\{i_1, i_2, \dots, i_n\} \in \mathscr{D} \cap \mathscr{N}\setminus d_2} \Big\|\sum_{j=1}^{n} \alpha_{i_j} x_{i_j}\Big\| <$$

$$< \sum_{i\in \mathscr{N}\setminus d_1} \frac{1}{2^i} \|\alpha_i x_i\| + \sup_{\{i_1, i_2, \dots, i_n\} \in \mathscr{D} \cap \mathscr{N}\setminus d_1} \Big\|\sum_{j=1}^{n} \alpha_{i_j} x_{i_j}\Big\| = \Big\|\sum_{i\in \mathscr{N}\setminus d_1} \alpha_i x_i\Big\|_*.$$

Thus $\{x_n\}$ is strictly coorthogonal in $\|\cdot\|_*$ and the proof is complete.

6. Characterization of unconditional bases

Let $\{y_n\}$ be a sequence in a Banach space E. Let $P_d = [y_i]_{i \in d}$ where $d = \{i_1, i_2, \dots, i_n\} \subset \mathscr{N}$, i.e., $d \in \mathscr{D}$ and $P = \bigcup_{d \in \mathscr{D}} P_d$. For $p \in S$, let $p = \sum_{i \in d} \alpha_i y_i$, then

$$s_{d'}^d(p) = \begin{cases} \sum_{i \in d'} \alpha_i x_i & \text{if } d' \subset d \\ p & \text{if } d \subset d' \end{cases}$$

$$r_{d'}^d(p) = p - s_{d'}^d(p).$$

Remark 6.1. $s^d_{d'}(p)$ is not defined whenever $d' \cap d \neq \emptyset$ and neither $d \subset d'$ or $d' \subset d$ hold.

Definition 6.1. A norm on E is a

(i) *weak CNTK-norm* relative to $\{y_n\}$ if for each polynomial $p \in P$, $p = \sum_{i \in d} \alpha_i y_i$ and for each $d' \subset d$, the polynomial $\sum_{i \in d'} \alpha_i y_i$ is the unique best coapproximation to p from $[y_i]_{i \in d'}$.

Remark 6.1. It should be noted here that analogous definitions of weak CNT-, weak CNK-norms coincide with that of weak CNTK-norm.

Theorem 6.1. *Let $\{y_n\}$ be a non-zero sequence in a Banach space E with norm $\|\cdot\|$. The norm is a weak CNTK-norm relative to $\{y_n\}$ if and only if*

$$\sup_d \sup \{\|s_d(p)\| : p \in P, \|p\| \leqq 1\} = 1.$$

Proof. Similar to the proof of Theorem 4.1.

Remark 6.2. $s_d(p)$ will assume the role of $s^{d'}_d(p)$ whenever $p = \sum_{i \in d'} \alpha_i y_i \in P$.

Theorem 6.2. *The following statements about a sequence $\{y_n\}$ in a Banach space E with $[y_i]_{i \in \mathcal{N}} = E$ are equivalent:*

(i) *$\{y_n\}$ is an unconditional basis of E.*

(ii) *A weak CNTK-norm relative to $\{y_n\}$ can be introduced on E equivalent to the original norm on E.*

Proof. The proof is similar to that of Theorem 4.3.

7. Remarks

Let E be a Banach space with norm $\|\cdot\|$. A sequence $\{M_i\}$ of non-trivial subspaces of E is called a decomposition of E provided for each $x \in E$, there exists a unique sequence $\{x_i\}$ such that $x_i \in M_i$ and $\sum_{i=1}^{\infty} x_i = x$, the convergence being in the norm topology. It is also possible to define for each i, a projection $P_i: E \to M_i$ by $P_i(x) = x_i$. If each projection is continuous, then the pair $\{M_i, P_i\}$ is called a Schauder decomposition. The notions of a Schauder basis and a Schauder decomposition are almost similar in the view point of approximation theory. Best approximation and Schauder decompositions were studied by P. K. JAIN and K. AHMAD [4], [5], [6]. Hence the analogous results of best coapproximation and bases in Banach spaces can be carried over to Schauder decompositions. Though the results look different, the idea is the same. Therefore the analogous results, even though known to the authors, are not elaborated.

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Schmidtsche Umkehrbedingungen für Potenzreihenverfahren

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1. Einleitung

Es sei $\{p_n\}_{n=0}^{\infty}$ eine Folge nichtnegativer Zahlen mit $p_0>0$, für welche die Reihe

$$(1.1) \qquad p(x) := \sum_{n=0}^{\infty} p_n x^n$$

den Konvergenzradius 1 hat und

$$(1.2) \qquad P_n := p_0 + \ldots + p_n \to \infty$$

gilt. Die Folge $s=\{s_n\}_{n=0}^{\infty}$ komplexer Zahlen heißt *J_p-limitierbar zum Wert σ* (J_p-$\lim s_n=\sigma$), wenn die Reihe

$$(1.3) \qquad p_s(x) := \sum_{n=0}^{\infty} p_n s_n x^n$$

für $0<x<1$ konvergiert und $\lim\limits_{x\to 1-} p_s(x)/p(x)=\sigma$ gilt. Die Folge s heißt *M_p-limitierbar zum Wert σ* (M_p-$\lim s_n=\sigma$), wenn

$$(1.4) \qquad t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \to \sigma$$

gilt. Bekanntlich sind die Verfahren J_p und M_p (wegen (1.2)) permanent, d. h., sie limitieren jedes $s\in c$, dem Raum der konvergenten Folgen, zum Wert $\lim s_n$, und aus M_p-$\lim s_n=\sigma$ folgt stets J_p-$\lim s_n=\sigma$ (ISHIGURO [6]), aber nicht umgekehrt. Unter geeigneten zusätzlichen Bedingungen für die Folge s kann man jedoch von M_p-$\lim s_n=\sigma$ auf $\lim s_n=\sigma$, von J_p-$\lim s_n=\sigma$ auf M_p-$\lim s_n=\sigma$ oder sogar von J_p-$\lim s_n=\sigma$ auf $\lim s_n=\sigma$ zurückschließen. Solche Bedingungen heißen *Umkehr-* oder *Tauber-Bedingungen*. Etwas genauer werden wir eine derartige Bedingung, falls einer der eben genannten Fälle vorliegt, eine *Tauber-Bedingung* (TB) *vom*

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$M_p \to c$-Typ, *vom* $J_p \to M_p$-Typ bzw. *vom* $J_p \to c$-Typ nennen. Zum Beispiel ist für $p_n :\equiv 1$ (das *Potenzreihenverfahren*) J_p das *Abel-Verfahren* A. Hierfür gilt der folgende Satz von SCHMIDT [23; Theorem X und Theorem XI].

Satz S. *Die Bedingung* $\liminf (s_n - s_m) \geqq 0$ *(falls die* s_n *reell sind bzw.* $\lim (s_n - s_m) = 0$, *falls die* s_n *komplex sind) für* $n/m \to 1$ *mit* $n > m \to \infty$, *ist eine* TB *vom* $A \to c$-Typ.

Aus Satz S erhält man auf einfache Weise die beiden im nachfolgenden Satz enthaltenen Resultate von LITTLEWOOD [17] bzw. HARDY and LITTLEWOOD [4; Theorem 11]. (Wie stets im folgenden sei $s_{-1} := 0$ und $a_n := s_n - s_{n-1}$ für $n = 0, 1, \ldots$.)

Satz HL. *Die Bedingung* $na_n = O_L(1)$ *(falls die* s_n *reell sind bzw.* $na_n = O(1)$, *falls die* a_n *komplex sind) ist eine* TB *vom* $A \to c$-Typ.

Hauptziel der vorliegenden Arbeit ist die Verallgemeinerung der Sätze S und HL für eine große Klasse von Verfahren J_p (Sätze 3.9 und 4.1). Alle dem Verfasser bekannten Sätze mit Tauber-Bedingungen dieser Art sind in den Sätzen 3.9 bzw. 4.1 als Spezialfälle enthalten.

2. Bezeichnungen und Definitionen

O, O_L und o sind die bekannten Landau-Symbole (vgl. [3]). Wenn nichts Besonderes gesagt ist, soll der Folgenindex von 0 an laufen, und statt $x_n \to \xi$ für $n \to \infty$ schreiben wir nur $x_n \to \xi$. Die Funktionen p und p_s sowie die Folgen $\{P_n\}$ und $\{t_n\}$ haben stets die Bedeutung wie in (1.1) bis (1.4).

Das zum Abel-Verfahren A gehörende Verfahren M_p ist das *Cesàro-Verfahren* der Ordnung 1. Für $\alpha > 0$ und $p_n := \binom{n+\alpha-1}{n}$ ist J_p das *verallgemeinerte Abel-Verfahren* A_α (A_1 ist das Abel-Verfahren). Für $p_n := (n+1)^{-1}$ sind J_p und M_p die *logarithmischen Verfahren* L und l. Für $\alpha > 0$ und $p(x) := [-x^{-1} \ln (1-x)]^\alpha$ ergeben sich die *verallgemeinerten logarithmischen Verfahren* L_α und l_α. In einer anderen Richtung werden L und l von PHILLIPS [20] verallgemeinert. (Bei diesen Verfahren gilt $p_n = 0$ für $0 \leqq n < n_0$. Wir betrachten die M_p-Transformierte $\{t_n\}$ dann nur für $n \geqq n_0$.)

3. Schmidtsche Bedingungen für $\{s_n\}$

Es sei $r \geqq 0$ eine reelle Konstante. Wir betrachten die Bedingung

$$\text{(3.1(r))} \qquad \liminf (s_n - s_m) \geqq -r \quad \text{für} \quad P_n/P_m \to 1 \quad (n > m \to \infty)$$

sowie, falls die s_n komplex sind, die Bedingung

$$\text{(3.2(r))} \qquad \limsup |s_n - s_m| \leqq r/\sqrt{2} \quad \text{für} \quad P_n/P_m \to 1 \quad (n > m \to \infty)$$

und beweisen zunächst einige Sätze über M_p.

3.1. Satz. *Gilt*

$$\text{(3.3)} \qquad P_{n+1}/P_n \to 1$$

und (3.1(r)) (*bzw.* (3.2(r))), *so folgt aus* M_p-$\lim s_n = \sigma$ *stets* $\limsup |s_n - \sigma| \leqq r$.

Beweis. Es genügt, den Satz für reelle s_n und $\sigma = 0$ zu beweisen. Wegen (3.1(r)) gibt es Zahlen $\delta > 0$ und $M \in \mathbf{N}$ so, daß

$$\text{(3.4)} \qquad s_n - s_m \geqq -r - \varepsilon \quad \text{für} \quad P_n/P_m \leqq 1 + \delta \quad \text{und} \quad n > m > M$$

gilt. Darüber hinaus können n und m wegen $P_n \to \infty$ und (3.3) so gegen ∞ streben, daß für ein geeignetes $M_0 > M$

$$\text{(3.5)} \qquad 1 + \delta/2 \leqq P_n/P_m \leqq 1 + \delta \quad \text{für} \quad m > M_0$$

gilt. Damit folgt für $n > m > M_0$ nach (3.4) zunächst

$$P_n t_n - P_m t_m = \sum_{\nu = m+1}^{n} p_\nu s_\nu \leqq (s_n + r + \varepsilon)(P_n - P_m),$$

also

$$\frac{P_n}{P_m} t_n - t_m \leqq (s_n + r + \varepsilon)\left(\frac{P_n}{P_m} - 1\right),$$

woraus sich wegen $t_n \to 0$ und (3.5) die Ungleichung $\liminf (s_n + r + \varepsilon) \geqq 0$, also, da $\varepsilon > 0$ beliebig gewählt war, $\liminf s_n \geqq -r$ ergibt. Entsprechend folgt $\limsup s_n \leqq r$ aus $P_n t_n - P_m t_m \geqq (s_m - r - \varepsilon)(P_n - P_m)$, insgesamt also $\limsup |s_n| \leqq r$.

3.2. Korollar. *Gilt* (3.3), *so ist* (3.1(0)) (*bzw.* (3.2(0))) *eine* TB *vom* $M_p \to$ $\to c$-Typ.

Kwee [15; Lemma 1] formuliert den reellen Teil von Korollar 3.2 ohne die Voraussetzung (3.3). Das Beispiel

$$p_{2k} := 2^k, \quad p_{2k+1} := 0, \quad s_{2k} := 0, \quad s_{2k+1} := 1 \quad (k = 0, 1, \ldots)$$

zeigt jedoch, daß man auf (3.3) nicht verzichten kann: Für die so definierte divergente Folge $\{s_n\}$ ist offensichtlich M_p-$\lim s_n = 0$. Ferner ist $P_{2k} = P_{2k+1} = 2^{k+1} - 1$

für $k=0, 1, \dots$, also gilt (3.3) nicht, wohl aber $(3.1(0))$, denn $P_n/P_m \to 1$ $(n>m\to\infty)$ ist genau dann erfüllt, wenn von einer Stelle an $n=m+1=2k+1$ gilt, und es ist $s_n-s_m=1$ für $n=m+1=2k+1$.

Eine im folgenden ständig wiederkehrende Bedingung ist

$$(3.6)\qquad 1\leqq \frac{P_n}{P_m}\to 1 \quad \text{für} \quad 1<\frac{n}{m}\to 1 \quad (m\to\infty).$$

Gilt (3.6), so folgt aus $(3.1(\mathrm{r}))$ offensichtlich

$$(3.7(\mathrm{r}))\qquad \liminf\,(s_n-s_m)\geqq -r \quad \text{für} \quad n/m\to 1,$$

und aus $(3.2(\mathrm{r}))$ ergibt sich

$$(3.8(\mathrm{r}))\qquad \limsup\,|s_n-s_m|\leqq r/\sqrt{2} \quad \text{für} \quad n/m\to 1.$$

Unter der Voraussetzung (3.6) sind die Bedingungen $(3.7(\mathrm{r}))$ und $(3.8(\mathrm{r}))$ also nicht stärker als $(3.1(\mathrm{r}))$ bzw. $(3.2(\mathrm{r}))$. Aus (3.6) folgt auch (3.3), und der Beweis von Satz 3.1 zeigt:

3.3. Satz. *Gilt* (3.6) *und* $(3.7(\mathrm{r}))$ $(bzw.\ (3.8(\mathrm{r})))$, *so folgt aus* M_p-$\lim s_n=\sigma$ *stets* $\limsup|s_n-\sigma|\leqq r$.

3.4. Korollar. *Gilt* (3.6), *so ist* $(3.7(0))$ $(bzw.\ (3.8(0)))$ *eine* TB *vom* $M_p\to$ $\to c$-Typ.

Spezialfälle der Korollare 3.2 und 3.4 finden sich bei Kwee [12; Lemma 3] (für $M_p=l$) und [14; Theorem 4], Phillips [20, Lemma 3] und Jakimovski and Tietz [8; Theorem 6.1].

Der Beweis von Satz 3.1 zeigt, daß man in den Sätzen 3.1 und 3.3 immer noch $s_n=O(1)$ erhält, wenn man die Voraussetzung M_p-$\lim s_n=\sigma$ zu $t_n=O(1)$ abschwächt. Zum Beweis des nächsten Satzes benötigen wir eine Variante eines bekannten Satzes von Vijayaraghavan.

3.5. Lemma (vgl. Rangachari [21; Lemma 1]). *Ist* $c_n(t):=p_n e^{-n/t}/p(e^{-1/t})$ *für* $n=0, 1, \dots$ *und* $t>0$, *gibt es eine positive, streng monoton gegen* ∞ *strebende Funktion* $f\colon (0,\infty)\to\mathbf{R}$ *mit* $f(t+1)-f(t)\to 0$ *für* $t\to\infty$,

$$(3.9)\qquad \sum_{n=0}^{M} c_n(t)\to 0 \quad \textit{für} \quad f(t)-f(M)\to\infty \quad (t>M\to\infty),$$

$$(3.10)\qquad \sum_{n=N}^{\infty} c_n(t)[f(n)-f(N)]\to 0 \quad \textit{für} \quad f(N)-f(t)\to\infty \quad (N>t\to\infty),$$

und existieren, bei vorgegebener Folge $\{s_n\}$, *zur Funktion*

$$s(t):=s_n \quad \textit{für} \quad n\leqq t<n+1 \quad (n=0, 1, \dots)$$

Konstanten $a>0$, $b>0$ *mit*

(3.11) $$s(t)-s(u)>-a[f(t)-f(u)]-b \quad \textit{für} \quad t>u>0,$$

so folgt aus $p_s(x)/p(x)=O(1)$ *für* $x\to 1-$ *auch* $s_n=O(1)$.

3.6. Satz. *Gilt* (3.3),

(3.12) $$P_{2n}=O(P_n)$$

und (3.1 (r)) (*bzw.* (3.2(r))), *so folgt aus* $p_s(x)/p(x)=O(1)$ *für* $x\to 1-$ *auch* $s_n=O(1)$.

Beweis. Es genügt, den Satz für reelle s_n zu beweisen. Sei also $\{s_n\}$ eine reelle Folge mit (3.1(r)) und $p_s(x)/p(x)=O(1)$ für $x\to 1-$. Wir definieren $c_n(t)$ und $s(t)$ wie in Lemma 3.5. Ferner sei

(3.13) $$f(t):=\ln P_{[t]} \quad \text{für} \quad 0<t<\infty.$$

Dann ist f: $(0,\infty)\to\mathbf{R}$ eine positive (wir dürfen ohne Einschränkung der Allgemeinheit $p_0>1$ annehmen), monoton (aber nicht streng monoton) gegen ∞ strebende Funktion mit $f(t+1)-f(t)\to 0$ für $t\to\infty$, die, wie gleich gezeigt wird, auch (3.9) bis (3.11) erfüllt. Zu f gibt es dann eine Funktion g, die neben den soeben aufgezählten Eigenschaften von f auch noch streng monoton ist und damit alle Eigenschaften besitzt, die in Lemma 3.5 (von f) verlangt werden. Demnach genügt es zum Beweis des Satzes noch zu zeigen, daß f aus (3.13) die Bedingungen (3.9) bis (3.11) erfüllt. Zu (3.9): Es gilt

(3.14) $$f(t)-f(M)\to\infty \Rightarrow P_{[t]}/P_M\to\infty.$$

Ferner ist nach Borwein and Kratz [1; Lemma 3(i)]

$$P_{[t]}\leqq \inf\{p(x)/x^{[t]}:\ x\in(0,1)\},$$

also insbesondere $P_{[t]}\leqq p(e^{-1/t})e$ für jedes $t>0$, woraus sich, zusammen mit (3.14),

$$\sum_{n=0}^{M} c_n(t)\leqq\frac{P_M}{p(e^{-1/t})}=\frac{P_M}{P_{[t]}}\cdot\frac{P_{[t]}}{p(e^{-1/t})}=o(1) \quad \text{für} \quad f(t)-f(M)\to\infty\ (t>M\to\infty)$$

ergibt. Zu (3.10): Aus

$$f(N)-f(t)\to\infty \Rightarrow P_N/P_{[t]}\to\infty$$

folgt wegen (3.12)

(3.15) $$f(N)-f(t)\to\infty \Rightarrow N/t\to\infty.$$

Ferner ist (vgl. [28; Lemma 2b)])

(3.16) $$p(e^{-1/N})=O(P_N) \quad \text{für} \quad N\to\infty.$$

Gilt nun $f(N)-f(t)\to\infty$ $(N>t\to\infty)$, so wählen wir nach (3.15) ein $t_0>0$ so, daß $N>2t$ für $t>t_0$ gilt. Wir erhalten dann für $t>t_0$

$$\sum_{n=N}^{\infty} c_n(t)\,[f(n)-f(N)] = \frac{1}{p(e^{-1/t})}\sum_{n=N+1}^{\infty} p_n e^{-n/t}\ln\frac{P_n}{P_N} \leqq$$

$$\leqq \frac{1}{P_N\, p(e^{-1/t})}\sum_{n=N+1}^{\infty} p_n e^{-n/t}(P_n-P_N) = \frac{1}{P_N\, p(e^{-1/t})}\sum_{\nu=N+1}^{\infty} p_\nu \sum_{n=\nu}^{\infty} p_n e^{-n/t} =$$

$$= \frac{1}{P_N\, p(e^{-1/t})}\sum_{\nu=N+1}^{\infty} p_\nu e^{-\nu/t}\sum_{n=\nu}^{\infty} p_n e^{-(n-\nu)/t} \leqq \frac{p(e^{-1/N})}{P_N\, p(e^{-1/t})}\sum_{\nu=N+1}^{\infty} p_\nu e^{-\nu/t} e^{\nu/N} =$$

$$= \frac{p(e^{-1/N})}{P_N\, p(e^{-1/t})}\sum_{\nu=N+1}^{\infty} p_\nu e^{-\nu/2t} e^{-\nu(N-2t)/2Nt} \leqq \frac{p(e^{-1/N})}{P_N\, p(e^{-1/t})}\sum_{\nu=N+1}^{\infty} p_\nu e^{-\nu/2t} e^{1-N/2t} \leqq$$

$$\leqq \frac{p(e^{-1/N})}{P_N}\cdot\frac{p(e^{-1/2t})}{p(e^{-1/t})}e^{1-N/2t} = o(1) \quad \text{für} \quad f(N)-f(t)\to\infty \quad (N>t\to\infty),$$

wenn wir (3.15) und (3.16) beachten sowie $p(x)/p(x^2)=O(1)$ für $x\to 1-$, was aus $P_n\to\infty$ und (3.12) folgt (vgl. Kratz and Stadtmüller [11; Lemma 2, proof of (vii)⇒(viii)]). Zu (3.11): Wegen (3.3) und (3.1(r)) gibt es nach Mikhalin [18; Lemma 2] positive Konstanten a und b mit

$$s(t)-s(u) \geqq -a\ln(P_{[t]}/P_{[u]})-b \quad \text{für} \quad t>u>0,$$

womit alles gezeigt ist.

Spezialfälle von Satz 3.6 finden sich bei Davydov [2; Sätze 6 und 7] (für $J_p=A$), Jeyarajan [9; Theorem 1] (für $J_p=A_\alpha$. In diesem Fall ist $P_n/P_m\to 1$ äquivalent zu $n/m\to 1$) und Mikhalin [18; Satz 2].

In Satz 3.6 darf (3.3)∧(3.12) durch (3.6) ersetzt werden, denn (3.6) ist äquivalent zu (3.12)∧

$$\lim_{t\to 1-}\liminf_{n\to\infty}\frac{1}{P_n}\sum_{tn<k\leqq n} p_k = 0 \tag{3.17}$$

und ist auch äquivalent zu (3.12)∧

$$\lim_{t\to 1+}\limsup_{n\to\infty}\frac{1}{P_n}\sum_{n<k\leqq tn} p_k = 0 \tag{3.18}$$

(vgl. Stadtmüller and Trautner [25]). Ferner wird (3.6) durch

$$np_n = O(P_n) \tag{3.19}$$

impliziert, darf also als Voraussetzung immer durch die stärkere, aber oft bequemere Bedingung (3.19) ersetzt werden.

Da unter der Voraussetzung (3.6) die Bedingung $s_n=O(1)$ eine TB vom $J_p\to$ $\to M_p$-Typ ist (TietzT und rautner [29; Korollar 4. 2]),erhalten wir aus Satz 3.6 unmittelbar:

3.7. Satz. *Gilt* (3.6), *so ist* (3.1(r)) (*bzw.* (3.2(r))) *eine* TB *vom* $J_p \to M_p$-Typ.

Spezialfälle von Satz 3.7 finden sich bei KOKHANOVSKII [10; Satz 5] (für $L \to l$) und TESLENKO [27; Satz 4] (für $L_\alpha \to l_\alpha$).

Aus den Sätzen 3.1 und 3.7 erhalten wir das folgende Analogon zu Satz 3.1 für J_p.

3.8. Korollar. *Gilt* (3.6) *und* (3.1(r)) (*bzw.* (3.2(r))), *so folgt aus* J_p-$\lim s_n = \sigma$ *stets* $\limsup |s_n - \sigma| \leqq r$.

Spezialfälle von Korollar 3.8 finden sich bei DAVYDOV [2; Sätze 6 und 7] (für $J_p = A$. Nach Davydov gilt für $J_p = A$ auch im komplexen Fall $\limsup |s_n - \sigma| \leqq \leqq r/\sqrt{2}$), KOKHANOVSKII [10; Satz 6] (für $J_p = L$), TESLENKO [27; Satz 5] (für $J_p = L_\alpha$) und MIKHALIN [18; Satz 2]. Für $r=0$ erhalten wir aus Korollar 3.8 schließlich die in der Einleitung angekündigte Verallgemeinerung von Satz S:

3.9. Satz. *Gilt* (3.6), *so ist* (3.1(0)) (*bzw.* (3.2(0))) *eine* TB *vom* $J_p \to c$-Typ.

Neben Satz S (vgl. auch VIJAYARAGHAVAN [30]) enthält Satz 3.9 als Spezialfälle Resultate von LANDAU [16; § 3] (für $J_p = A$), KWEE [12; Theorem A] (für $J_p = L$), [14; Theorem 8] (für $J_p = L_\alpha$) und [15; Theorem A]. (Die dortige Voraussetzung $p(x)/p(x^2) \to 1$ für $x \to 1-$ impliziert $P_{2n}/P_n \to 1$ (vgl. [29; Nr. 5]) und damit (3.6)), JEYARAJAN [9; Theorem 4] (für $J_p = A_\alpha$), PHILLIPS [20], SONI [24; Theorem 2] und JAKIMOVSKI and TIETZ [8; Theorem 6.3].

Die durch den Vergleich mit Korollar 3.4 nahegelegte Frage, ob in Satz 3.9 die Bedingungen (3.1(0)) und (3.2(0)) durch (3.7(0)) bzw. (3.8(0)) ersetzt werden dürfen, muß offen bleiben.

4. Umkehrbedingungen vom O_L- und O-Typ

Neben den in der Einleitung genannten generellen Voraussetzungen verlangen wir jetzt $p_n > 0$ für $n = 0, 1, \ldots$. Als einfache Folgerung aus Satz 3.9 erhalten wir die angekündigte Verallgemeinerung von Satz HL:

4.1. Satz. *Gilt* (3.6), *so ist* $P_n a_n = O_L(p_n)$ (*bzw.* $P_n a_n = O(p_n)$) *eine* TB *vom* $J_p \to c$-Typ.

Beweis. Es genügt zu zeigen, daß (3.1(0)) aus $P_n a_n = O_L(p_n)$ folgt. Wählen wir $K > 0$ mit $a_n > -K p_n / P_n$ für $n = 0, 1, \ldots$, so gilt

$$s_n - s_m > -K \sum_{\nu=m+1}^{n} \frac{p_\nu}{P_\nu} \geqq -K \left(\frac{P_n}{P_m} - 1 \right) \quad \text{für} \quad n > m,$$

woraus sich (3.1(0)) ergibt.

Neben Satz HL enthält Satz 4.1 als Spezialfälle Resultate von Ishiguro [5] (für $J_p=L$ und o statt O), Rangachari and Sitaraman [22; Theorems I(A) und I(L)] (für $J_p=A_\alpha$ und $J_p=L$), Ishiguro [6] und [7], Štěpánek [26], Phillips [20], Kokhanovskii [10; Korollar 1] (für $J_p=L$), Jakimovski and Tietz [8; Theorems 5.2 bis 5.4, 5.2* und 5.4*], Tietz und Trautner [29; Satz 4.4] und Kratz and Stadtmüller [11; Theorem 2]. Insbesondere ist damit auch [8; Theorem 5.1], dessen Beweis einen Fehler enthält, bewiesen und gleichzeitig verallgemeinert.

Da J_p-$\lim s_n=\sigma$ stets J_P-$\lim t_n=\sigma$ impliziert (wobei J_P das durch die Folge $\{P_n\}$ definierte Potenzreihenverfahren sein soll), ergibt sich mit

$$Q_n := P_0+\ldots+P_n \tag{4.1}$$

aus Satz 4.1 unmittelbar:

4.2. Korollar. *Gilt*

$$1 \leqq \frac{Q_n}{Q_m} \to 1 \quad \textit{für} \quad 1 < \frac{n}{m} \to 1 \quad (m \to \infty), \tag{4.2}$$

so ist $Q_n(t_n-t_{n-1})=O_L(P_n)$ (*bzw.* $Q_n(t_n-t_{n-1})=O(P_n)$) *eine* TB *vom* $J_p \to M_p$-Typ.

5. Schmidtsche Bedingungen für $\{t_n\}$

An zwei Beispielen wollen wir zeigen, wie die Ergebnisse aus Nr. 3 weiter ausgebaut werden können. Dazu betrachten wir in Analogie zu (3.1(r)) und (3.2(r)) jetzt die Bedingungen

$$\liminf (t_n-t_m) \geqq -r \quad \text{für} \quad Q_n/Q_m \to 1 \quad (n>m\to\infty) \tag{5.1(r)}$$

und

$$\limsup |t_n-t_m| \leqq r/\sqrt{2} \quad \text{für} \quad Q_n/Q_m \to 1 \quad (n>m\to\infty) \tag{5.2(r)}$$

(mit Q_n aus (4.1)). Diese sind beziehentlich nicht schwächer als die in Analogie zu (3.7(r)) und (3.8(r)) gebildeten Bedingungen

$$\liminf (t_n-t_m) \geqq -r \quad \text{für} \quad n/m \to 1 \quad (n>m\to\infty) \tag{5.3(r)}$$

und

$$\limsup |t_n-t_m| \leqq r/\sqrt{2} \quad \text{für} \quad n/m \to 1 \quad (n>m\to\infty), \tag{5.4(r)}$$

denn wegen

$$\frac{n}{m}-1=\frac{n-m}{m}\frac{P_m}{P_m} \leqq \frac{P_{m+1}+\ldots+P_n}{mP_n} = \frac{Q_n-Q_m}{Q_m}\frac{Q_m}{mP_m} \leqq \frac{Q_n}{Q_m}-1 \quad \text{für} \quad n>m>0$$

folgt $n/m \to 1$ aus $Q_n/Q_m \to 1$.

Unter der Voraussetzung (4.2) ist (5.1(r)) äquivalent zu (5.3(r)) und (5.2(r)) ist äquivalent zu (5.4(r)). Beachten wir noch, daß J_P-lim $t_n=\sigma$ aus J_p-lim $s_n=\sigma$ folgt, so erhalten wir aus Satz 3.9:

5.1. Satz. *Gilt* (4.2), *so ist* (5.3(0)) (*bzw.* (5.4(0))) *eine* TB *vom* $J_p \to M_p$-Typ.

Spezialfälle von Satz 5.1 finden sich bei Kokhanovskii [10; Satz 1] (für $L \to l$) und Teslenko [27; Satz 1] (für $L_\alpha \to l_\alpha$).

In Satz 5.1 darf (4.2) durch (3.6) ersetzt werden, denn (3.6) impliziert (4.2): Aus (3.6) folgt nach Stadtmüller and Trautner [25] zunächst (3.12) und daraus

$$nP_n = O(Q_n) \tag{5.5}$$

(vgl. Mikhalin [19]). Wählen wir danach ein $K>0$ mit $nP_n/Q_n<K$ für $n=0, 1, \dots$, so gilt $K \cdot \ln(n/m)<1$ für hinreichend nahe bei 1 liegende Quotienten n/m. Für derartige $n>m>0$ gilt ferner

$$\frac{Q_n}{Q_m} = 1+\frac{1}{Q_m}\sum_{k=m+1}^{n}\frac{kP_k}{Q_k}\frac{Q_k}{k} \leqq 1+K\frac{Q_n}{Q_m}\ln\frac{n}{m},$$

also $Q_n/Q_m \leqq [1-K\cdot\ln(n/m)]^{-1}$, woraus (4.2) folgt.

5.2. Korollar. *Gilt* (3.6), *so ist* (5.3(0)) (*bzw.* (5.4(0))) *eine* TB *vom* $J_p \to$ $\to M_p$-Typ.

Spezialfälle von Korollar 5.2 finden sich bei Kwee [13; Theorem 6] (für $L \to l$. Aus Kwees Bedingung

$$\liminf (t_n - t_m) \geqq 0 \quad \text{für} \quad P_n/P_m \to 1 \quad (n > m \to \infty) \tag{5.6}$$

folgt wegen (3.6) nämlich (5.3(0))) und bei Mikhalin [18; Satz 1].

Zum Vergleich von Korollar 5.2 mit Satz 3.7 sei noch bemerkt, daß die Bedingungen (5.3(0)) und (5.4(0)) nicht stärker sind als (3.1(r)) bzw. (3.2(r)). Nach Mikhalin [18; Lemma 3] folgt nämlich aus (3.3) und (3.1(r)) schon (5.6) und daraus (5.3(0)) wegen (3.6).

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Sur l'unicité de la décomposition de Kato généralisée

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1. Introduction et préliminaires

Pour un opérateur (linéaire) fermé A dans un espace de Hilbert H on désignera par $D(A)$ le domaine, $R(A)$ l'image, $N(A)$ le noyau. Pour un couple (M, N) de sous-espaces fermés de $H, M+N$ désignera leur somme vectorielle; la notation $M \oplus N$ sera réservée au cas où, en plus, $M \cap N = \{0\}$. La restriction de A à M sera notée $A|M$.

T. Kato [1, Théorème 4] a montré que si A est *semi-Fredholm*, il existe (M, N) tel que

(a) $H = M \oplus N$,

(b) $A(M \cap D(A)) \subseteqq M$ et, en posant $A_0 = A|M$, $R(A_0)$ est fermé et $N(A_0^n) \subseteqq R(A_0)$ pour $\forall n \geqq 0$,

(c) $A(N) \subseteqq N \subseteqq D(A)$ et $A|N$ est *nilpotent* de degré d fini.

Cette décomposition de H est appelée *décomposition de Kato* associée à A. Kato a aussi remarqué que pour A *semi-Fredholm* telle décomposition de H est unique à un isomorphisme près.

J. P. Labrousse [2] a caractérisé tous les opérateurs qui admettent telle décomposition; ces opérateurs sont appelés *quasi-Fredholm.*

L'auteur a étudié [5] les opérateurs A qui admettent une *décomposition du type de Kato* où la condition (c) est remplacée par:

$$\text{(c')} \qquad A(N) \subseteqq N \subseteqq D(A), \quad \text{et} \quad A|N \text{ est } \textit{quasi-nilpotent}.$$

Dans ce cas la décomposition (M, N) est appelée *décomposition de Kato généralisée* (D.K.G.) associée à A, et les opérateurs qui admettent telles décompositions sont appelés *pseudo-Fredholm.*

Si l'on peut choisir $M = H$ $(N = \{0\})$, A est dit *régulier* (voir [4], [6]).

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Pour des exemples et propriétés des opérateurs *pseudo-Fredholm* voir [5]. Disons toutefois que si l'on note par Φ, $s\Phi$, $q\Phi$, $p\Phi$ les ensembles des opérateurs de *Fredholm, semi-Fredholm, quasi-Fredholm* et *pseudo-Fredholm,* selon les cas, chacun de ces ensembles est strictement contenu dans le suivant. Remarquons aussi que l'ensemble des opérateurs *quasi-nilpotents* est inclus strictement dans l'ensemble des opérateurs *pseudo-Fredholm.*

Dans la suite on notera $K(A)$ le *coeur analytique* de A défini par:

$$K(A) = \{u\in H;\ \exists a>0,\ \forall n>0\ \exists v_n\in D(A) \text{ tels que } (1)\ v_0=u \text{ et } Av_{n+1}=v_n,\ (2)\ \|v_n\|\leqq a^n\|u\|\ \forall n\geqq 0\}.$$

On notera aussi $H_0(A)$ la *partie quasi-nilpotente* de A, définie par:

$$H_0(A) = \{u\in D^\infty(A);\ \lim_{n\to\infty}\|A^nu\|^{1/n}=0\} \quad \text{où} \quad D^\infty(A)=\bigcap_{n\geqq 0} D(A^n).$$

Remarque 1.1 ([3], [4]).

(a) $K(A)$ et $H_0(A)$ sont des sous-espaces de H non nécessairement fermés.

(b) $A(K(A)\cap D(A))=K(A)$ et $A(H_0(A))\subseteq H_0(A)\subseteq D(A)$.

(c) A *quasi-nilpotent* $\Rightarrow K(A)=K(A^*)=\{0\}$.

(d) A *quasi-nilpotent* $\Leftrightarrow H_0(A)=H$.

(e) Si A est *régulier,* alors $\overline{H_0(A)}=\overline{\bigcup_{n\geqq 0} N(A^n)}\subseteq K(A)$.

Théorème 1.2. *Si A est un opérateur fermé, les conditions suivantes sont équivalentes:*

(i) *A régulier et $H_0(A)$ fermé,*

(ii) *$R(A)$ fermé et $H_0(A)=\{0\}$,*

(iii) *$R(A)$ fermé et $N(A)=\{0\}$,*

(iv) *A régulier et $\bigcup_{n\geqq 0} N(A^n)$ fermé.*

Démonstration.

(i)$\Rightarrow$(ii) voir [4, Théorème 2.11].

(ii)$\Rightarrow$(iii) évident car $N(A)\subseteq H_0(A)$.

(iii)$\Rightarrow$(iv) $N(A)=\{0\}$ implique $\forall n\geqq 0$ $N(A^n)=\{0\}$, d'où $\bigcup_{n\geqq 0} N(A^n)=\{0\}$ et donc fermé.

(iv)$\Rightarrow$(i) *A régulier* et (iv) impliquement $\overline{H_0(A)}=\overline{\bigcup_{n\geqq 0} N(A^n)}=\bigcup_{n\geqq 0} N(A^n)\subseteq H_0(A)$ donc $\overline{H_0(A)}=H_0(A)$, d'où (i).

Théorème 1.3 (voir [4, Théorème 1.6]). *Si A est un opérateur fermé, alors les conditions suivantes sont équivalentes:*

(i) *$\lambda\in\sigma(A)$ est isolé dans $\sigma(A)$ (spectre de A)*

(ii) *$H=K(A-\lambda I)\oplus H_0(A-\lambda I)$ et $H_0(A-\lambda I)\Leftrightarrow\{0\}$.*

2. Quelques propriétés des opérateurs pseudo-Fredholm

Lemme 2.1. *Soient* $A \in p\Phi$ *et* (M, N) *une* D.K.G. *associée à* A. *Soient* P_M *et* P_N *respectivement les projections sur* M *et* N *suivant la décomposition* $H = M \oplus N$. *Alors on a:*

(a) $AP_M \in q\Phi(1)$ (*si* $N \neq \{0\}$ *et* $M \neq \{0\}$),

(b) $K(A) = K(AP_M)$ *est fermé,*

(c) $H_0(A) = H_0(AP_M)$.

Démonstration.

(a) $N \subseteqq N(AP_M)$ implique que la restriction de AP_M à N est *nilpotente de degré* 1. d'autre part la restriction de AP_M à M est égale à $A|M$ et donc *régulière*.

(b) Il suffit de démontrer que $K(A) \subseteqq K(AP_M)$. Soient $u \in K(A)$ et $a > 0$, $\{v_n\}_{n>0}$ de la définition de $K(A)$, alors $u = P_M u + P_N u = A^n v_n = (AP_M)^n v_n + (AP_N)^n v_n$ $\forall n > 0$. Donc $\forall n > 0$ $P_M u = (AP_M)^n v_n$ et $P_N u = (AP_N)^n v_n$. Montrons que $P_N u$ est nul. On a $\|P_N u\| \leqq \|(AP_N)^n\| \, \|v_n\| \leqq \varepsilon^n a^n \|u\|$ dès que n est assez grand car AP_N est quasi-nilpotent et par conséquent $\|P_N u\| = 0$ ou encore $P_N u = 0$. D'où $\forall n > 0$ $u = P_M u = (AP_M)^n v_n$ ce qui implique que $u \in \bigcap_{n \geqq 0} R[(AP_M)^n]$. D'après (a) et en utilisant [3, Théorème 1.5.4; Corollaire 2.1.6] on en déduit que $u \in K(AP_M)$; par conséquent $K(A) = K(AP_M)$ est fermé.

(c) $N \subseteqq H_0(A) \Rightarrow H_0(A) = H_0(A) \cap M + N = H_0(A|M) + N$. De même,

$$N \subseteqq N(AP_M) \subseteqq H_0(AP_M) \Rightarrow H_0(AP_M) = H_0(AP_M) \cap M + N =$$

$$= H_0(A|M) + N = H_0(A).$$

Remarque 2.2. Le Lemme 2.1 implique que $\forall (M, N)$ D.K.G. associée à A, $K(A) \subseteqq M$ et $N \subseteqq H_0(A)$.

Lemme 2.3. *Soient* $A \in p\Phi$ *et* (M, N) *une D.K.G. associée à* A. *Alors* $R(A) + N = A(M \cap D(A)) + N$ *est fermé dans* H.

Démonstration. $N \subseteqq D(A) \Rightarrow D(A) = M \cap D(A) + N \Rightarrow R(A) \subseteqq A(M \cap D(A)) + A(N) \subseteqq A(M \cap D(A)) + N$ donc $R(A) + N \subseteqq A(M \cap D(A)) + N$. L'autre inclusion étant évidente, on a l'égalité.

Montrons que $A(M \cap D(A)) + N$ est fermé. $H = M \oplus N \Rightarrow H$ est isomorphe à $M \times N$ et par conséquent $A(M \cap D(A)) + N$ est isomorphe à $A(M \cap D(A)) \times N$. Or $A(M \cap D(A))$ est fermé, d'où on déduit que $A(M \cap D(A)) \times N$ est fermé et donc $A(M \cap D(A)) + N$ est fermé.

Théorème 2.4. *Soit* A *un opérateur fermé de domaine dense dans* H. *Alors on a*

$$A \in p\Phi \Rightarrow A^* \in p\Phi.$$

Démonstration. Soient $A\in p\Phi$ et (M, N) une D.K.G. associée à A; on va montrer que $(N^\perp, M^\perp)$ est une D.K.G. associée à A^*. Remarquons d'abord que $H=M\oplus N\Rightarrow H=N^\perp\oplus M^\perp$.

(1) Montrons que $M^\perp\subseteq D(A^*)$. Soient $u\in M^\perp$ et $v\in D(A)$. On a $v=P_M v+P_N v$, $v\in D(A)$ et $P_N v\in D(A)$ impliquent que $P_M v\in D(A)$. Donc $(Av, u)=(AP_M v, u)+$ $+(AP_N v, u)$ or $A(M\cap D(A))\subseteq M\Rightarrow AP_M v\in M$. Et comme $u\in M^\perp$, on a $(AP_M v, u)=0$. Donc $(Av, u)=(AP_N v, u)$. $N\subseteq D(A)\Rightarrow AP_N$ est borné, donc $|(Av, u)|\leqq\|AP_N\|\,\|v\|\,\|u\|$ d'où $u\in D(A^*)$.

(2) $A^*(M^\perp)\subseteq M^\perp$ et $A^*(N^\perp\cap D(A^*))\subseteq N^\perp$ car $A(N)\subseteq N$ et $A(M\cap D(A))\subseteq M$.

(3) Montrons que $A^*|M^\perp$ est *quasi-nilpotent*. Pour cela, montrons d'abord que

$$(P_{M^\perp})^* = P_N.$$

Soient $u\in H$ et $v\in H$,

$$\begin{aligned}
\big((P_{M^\perp})^* u, v\big) &= (u, P_{M^\perp} v) = (P_M u+P_N u, P_{M^\perp} v) = \\
&= (P_M u, P_{M^\perp} v)+(P_N u, P_{M^\perp} v) = (P_N u, P_{M^\perp} v) = \\
&= \big(P_N u, (I-P_{N^\perp}) v\big) = (P_N u, v)-(P_N u, P_{N^\perp} v) = (P_N u, v)
\end{aligned}$$

Donc $\forall u\in H$, $\forall v\in H$ $\big((P_{M^\perp})^* u, v\big)=(P_N u, v)$, c'est-à-dire

$$(P_{M^\perp})^* = P_N \quad \text{et} \quad P_{M^\perp} = (P_N)^*.$$

D'autre part, $(AP_N)^*=(P_N AP_N)^*=P_{M^\perp}A^*P_{M^\perp}=A^*P_{M^\perp}$. Or AP_N *quasi-nilpotent* entraîne que $(AP_N)^*$ est *quasi-nilpotent*, donc $A^*P_{M^\perp}$ est *quasi-nilpotent* d'où $A^*|M^\perp$ est *quasi-nilpotent*.

(4) Reste à montrer que $A^*|N^\perp$ est *régulier*. Le même raisonnement utilisé dans (3) montre que $(P_{N^\perp})^*=P_M$. Par le Lemme 2.1 (a) on a $AP_M\in q\Phi(1)$ et d'après [2, Proposition 3.3.4] on a $(AP_M)^*\in q\Phi(1)$. Donc $A^*P_{N^\perp}\in q\Phi(1)$ et comme $R(A^*|N^\perp)=R(A^*P_{N^\perp})$ on en déduit que $R(A^*P_{N^\perp})$ est fermé. Donc pour montrer (4) il suffit de montrer que

$$N(A^*|N^\perp)\subseteq R(A^*P_{N^\perp}) \quad \text{car} \quad A^*P_{N^\perp}\in q\Phi(1).$$

$u\in N(A^*|N^\perp)=N(A^*)\cap N^\perp\Rightarrow u\perp(R(A)+N)$. Or d'après le Lemme 2.3, $R(A)+N=$ $=A(M\cap D(A))+N$ est fermé. Donc $u\in[A(M\cap D(A))+N]^\perp$. Or $N(AP_M)=$ $=N(A|M)+N\subseteq R(A|M)+N\Rightarrow[A(M\cap D(A))+N]^\perp\subseteq N(AP_M)^\perp$. Donc

$$u\in N(AP_M)^\perp = R\big((AP_M)^*\big)$$

qui est fermé car $(AP_M)^*\in q\Phi(1)$. Donc $u\in R(A^*P_{N^\perp})=R(A^*|N^\perp)$, d'où finalement $N(A^*|N^\perp)\subseteq R(A^*|N^\perp)$ et donc

$$(N^\perp, M^\perp) \text{ est une D.K.G associée à } A^*.$$

Proposition 2.5. *Soit* $A \in p\Phi$ *avec* $D(A)$ *dense dans* H. *Alors on a*

$$H_0(A)^{\perp} = K(A^*).$$

Démonstration. Elle se déduit du Lemme 2.1 et [3, Proposition 2.3.2].

3. Sur l'unicité de la décomposition de Kato généralisée

Proposition 3.1. *Soient* (M_1, N_1) *et* (M_2, N_2) *deux* D.K.G. *associées à* A, *alors:* $\forall i=1,2$ $\forall j=1,2$ (M_i, N_j) *est une* D.K.G. *associée à* A.

Démonstration. Par hypothèse on a M_i et N_j invariants par A. Notons $A|M_i$ et $A|N_j$ respectivement la restriction de A à M_i et N_j. $A|M_i$ est *régulier* et $A|N_j$ est *quasi-nilpotent*, donc il reste à montrer que $H=M_i \oplus N_j$ pour $i=1,2$ et $j=1,2$. Si $i=j$ alors le résultat est vrai par hypothèse, si non, d'après la Remarque 2.2 on a $K(A) \subseteqq M_i$ et $N_j \subseteqq H_0(A)$. Remarquons que $\forall i=1,2$ $M_i \cap \cap H_0(A) = K(A) \cap H_0(A)$. L'inclusion «$\supseteqq$» est évidente. Montrons l'autre inclusion: $M_i \cap H_0(A) = H_0(A|M_i) \subseteqq K(A) \Rightarrow M_i \cap H_0(A) \subseteqq K(A) \cap H_0(A)$. D'où l'égalité. Montrons maintenant que $M_i \cap N_j = \{0\}$. On a $M_i \cap N_j \subseteqq M_i \cap H_0(A) \subseteqq K(A) \subseteqq M_j$. Donc $M_i \cap N_j \subseteqq M_j \cap N_j = \{0\}$ car (M_j, N_j) est une D.K.G.

Il reste à montrer que $H=M_i+N_j$; pour cela montrons que $\forall j=1,2$ $K(A)+ +H_0(A)=N_j+K(A)$. «$\supseteqq$» est évident car $N_j \subseteqq H_0(A)$. Inversement, $H_0(A)= =M_j \cap H_0(A)+N_j \subseteqq K(A)+N_j$ d'où $\forall j=1,2$ $H_0(A) \subseteqq K(A)+N_j$ et finalement $K(A)+H_0(A) \subseteqq K(A)+N_j$. D'où l'égalité cherchée. Donc:

$$H = M_i+N_i \subseteqq M_i+H_0(A) = M_i+K(A)+H_0(A) = M_i+K(A)+N_j = M_i+N_j.$$

Donc:

$$\forall i = 1,2 \quad \text{et} \quad \forall j = 1,2, \quad H = M_i \oplus N_j.$$

Soit $S \in B(H)$. On dira que S commute avec A si $S(D(A))=D(A)$ et $\forall u \in D(A)$ on a $ASu=SAu$.

Théorème 3.2. *La* D.K.G. *est unique à un isomorphisme près commutant avec* A.

Démonstration. 1) Montrons que si (M, N) est une D.K.G. associée à A alors $\forall S \in B(H)$ inversible et commutant avec A, $(S(M), S(N))$ est une D.K.G. associée à A.

En effet, $H=S(H)=S(M)+S(N)$ et d'autre part si $u \in S(M) \cap S(N)$ alors $\exists v \in M$ et $\exists w \in N$ tels que $u=Sv=Sw$. Ce qui implique $v=w$ car S est injectif. Donc $v=w \in M \cap N=\{0\}$ d'où $u=0$. Donc $H=S(M) \oplus S(N)$ et $S(M)$, $S(N)$ sont fermés.

Pour l'invariance remarquons que

$$S(D(A)) = D(A) \Rightarrow S(N) \subseteq D(A) \quad \text{et} \quad S(M) \cap D(A) = S(M \cap D(A))$$

$$AS = SA \Rightarrow A(S(M) \cap D(A)) = A(S(M \cap D(A))) = S(A(M \cap D(A))) \subseteq S(M)$$

et

$$A(S(N)) = S(A(N)) \subseteq S(N).$$

Montrons que $A|S(N)$ est *quasi-nilpotent*. Tout d'abord on a $N \subseteq H_0(A)$ et S, borné, commute avec A, impliquent $S(N) \subseteq H_0(A)$. Comme $S(N)$ est fermé on en déduit que $A|S(N)$ est *quasi-nilpotent*.

Reste à montrer que $A_0 = A|S(M)$ est *régulier*. Pour cela montrons d'abord que $R(A_0) = A(S(M) \cap D(A))$ est fermé. Soit $u_n \in A(S(M) \cap D(A))$ tel que $u_n \to u$ dans H.

$u_n \in R(A_0) \Rightarrow \exists v_n \in M \cap D(A)$ tel que $u_n = ASv_n = SAv_n$. Or $S^{-1} \in B(H)$ donc $S^{-1}u_n \to S^{-1}u$ ce qui implique $Av_n \to S^{-1}u$. Et comme $A(M \cap D(A))$ est fermé on en déduit que $S^{-1}u \in A(M \cap D(A))$. Donc $\exists v \in M \cap D(A)$ tel que $S^{-1}u = Av$, d'où $u = SAv = ASv \in R(A_0)$.

Maintenant montrons que $\forall n \geqq 0$ $N(A_0^n) \subseteq R(A_0)$. Pour $u \in N(A_0^n)$ on a $u \in S(M)$ et $A^n u = 0$, donc $\exists v \in M \cap D(A)$ tel que $u = Sv$. Or, $A^n Sv = 0 \Rightarrow SA^n v = 0 \Rightarrow A^n v = 0$ donc $v \in N(A^n) \cap M = N((A|M)^n) \subseteq R(A|M)$ car $A|M$ est *régulier*. Donc $\exists w \in M \cap D(A)$ tel que $v = Aw$ d'où $u = Sv = SAw = ASw$ donc $u \in R(A_0)$.

2) Soient (M_1, N_1) et (M_2, N_2) deux D.K.G. associées à A, et soient P_{M_i}, P_{N_i} les projections sur M_i, N_i $(i = 1, 2)$ suivant la décomposition $H = M_i \oplus N_i$.

Posons $S = P_{M_2}P_{M_1} + P_{N_2}P_{N_1}$, il est clair que $S \in B(H)$. Vérifions que $S(D(A)) = D(A)$. Soit $u \in D(A)$. On a $u = P_{M_1}u + P_{N_1}u \Rightarrow P_{M_1}u \in D(A)$ et de même $P_{M_1}u = P_{M_2}P_{M_1}u + P_{N_2}P_{M_1}u$ et $P_{N_2}P_{M_1}u \in D(A) \Rightarrow P_{M_2}P_{M_1}u \in D(A)$. D'où $Su \in D(A)$ et donc

$$S(D(A)) \subseteq D(A).$$

Montrons l'inclusion inverse, On a $N_2 \subseteq D(A) \Rightarrow D(A) = M_2 \cap D(A) + N_2$. Soit $u \in D(A)$: alors $u = v + w$ avec $v \in M_2 \cap D(A)$ et $w \in N_2$. D'après la Proposition 3.1, $H = M_1 \oplus N_2$, donc $D(A) = M_1 \cap D(A) + N_2$ et $v = v_1 + v_2$ avec $v_1 \in M_1 \cap D(A)$ et $v_2 \in N_2$. Donc, $v_1 = P_{M_1}v_1$ et $v \in M_2$. D'où $v = P_{M_2}v = P_{M_2}P_{M_1}v_1 + P_{M_2}v_2$. Or, $v_2 \in N_2 \Rightarrow P_{M_2}v_2 = 0$. Donc:

$$(1) \qquad v = P_{M_2}P_{M_1}v_1 \quad \text{avec} \quad v_1 \in D(A).$$

De même, d'après la Proposition 3.1, $H = M_2 \oplus N_1$ donc $D(A) = M_2 \cap D(A) + N_1$. Or $w \in D(A)$ donc $w = w_1 + w_2$ avec $w_1 \in M_2 \cap D(A)$ et $w_2 \in N_1$. Donc $w = w_1 + P_{N_1}w_2$. Or, $w \in N_2 \Rightarrow w = P_{N_2}w = P_{N_2}w_1 + P_{N_2}P_{N_1}w_2 = P_{N_2}P_{N_1}w_2$ car $w_1 \in M_2$. Donc

$$(2) \qquad w = P_{N_2}P_{N_1}w_2 \quad \text{avec} \quad w_2 \in N_1 \subseteq D(A).$$

Alors (1) et (2)$\Rightarrow u=v+w=P_{M_2}P_{M_1}v_1+P_{N_2}P_{N_1}w_2$ avec $P_{N_2}P_{N_1}v_1=0$ et $P_{M_2}P_{M_1}w_2=0$. Donc $u=P_{M_2}P_{M_1}(v_1+w_2)+P_{N_2}P_{N_1}(w_2+v_1)=S(v_1+w_2)$ avec $v_1+w_2\in D(A)$ d'où $D(A)\subseteqq S(D(A))$ et par conséquent:

$$S(D(A)) = D(A). \tag{3}$$

Remarquons que le même raisonnement montre que $\forall u\in H$ $\exists(v_1+w_2)\in H$ tel que $u=S(v_1+w_2)$ donc S est surjectif. L'invariance de M_i et N_i (pour $i=1, 2$), et (3) impliquent que S commute avec A.

Montrons que S est injectif. Soit $u\in N(S)\Rightarrow Su=0\Rightarrow P_{M_2}P_{M_1}u=-P_{N_2}P_{N_1}u\Rightarrow P_{M_2}P_{M_1}u=0$ et $P_{N_2}P_{N_1}u=0$. D'où $P_{M_1}u\in N_2$ et $P_{N_1}u\in M_2$. Ce qui entrîane $P_{M_1}u\in N_2\cap M_1$ et $P_{N_1}u\in M_2\cap N_1$. Or d'après la Proposition 3.1 on a $N_2\cap M_1=M_2\cap N_1=\{0\}$ d'où $P_{M_1}u=0$ et $P_{N_1}u=0$ donc $u=P_{M_1}u+P_{N_1}u=0$ ou encore $N(S)=\{0\}$.

Finalement montrons que $S(M_1)=M_2$ et $S(N_1)=N_2$. Par définition de S on a $S(M_1)\subseteqq M_2$ et $S(N_1)\subseteqq N_2$. Soit maintenant $u\in M_2$; puisque $H=M_1\oplus N_2$ et, on a $u=v+w$ avec $v\in M_1$ et $w\in N_2$ donc $u=P_{M_2}u=P_{M_2}P_{M_1}v+P_{M_2}w$. Or, $w\in N_2\Rightarrow$ $\Rightarrow P_{M_2}w=0$; donc $u=P_{M_2}P_{M_1}v$, et $v\in M_1\Rightarrow P_{N_1}v=0$. Par conséquent $u=S(v)$ avec $v\in M_1$ donc $M_2\subseteqq S(M_1)$. Le même raisonnement montre que $N_2\subseteqq S(N_1)$ et donc que $S(M_1)=M_2$ et $S(N_1)=N_2$. Remarquons que les deux dernières égalités impliquent que S est surjectif. En effet $S(H)=S(M_1)+S(N_1)=M_2+N_2=H$, d'où la fin de la démonstration du Théorème.

Soit (M, N) une D.K.G. associée à A. On dira qu'elle est orthogonale si $M=N^\perp$.

Proposition 3.3 *Si* $A\in p\Phi$, *A admet au plus une* D.K.G. *orthogonale.*

Démonstration. Supposons qu'il existe deux D.K.G. orthogonales (M_1, N_1) et (M_2, N_2); donc $M_1=N_1^\perp$ et $M_2=N_2^\perp$. Montrons que $N_1=N_2$.

Soit $u\in N_2\subseteqq H=M_1\oplus N_1$, $u=u_1+u_2$ avec $u_1\in M_1$ et $u_2\in N_1$. Alors $u-u_2=$ $=u_1\in M_1\cap H_0(A)\subseteqq K(A)\subseteqq M_2=N_2^\perp$. Donc $u_2\perp u_1$ et $u-u_2\perp u$ (car $u\in N_2$). Donc $\|u_1\|^2=(u_1, u_1)=(u-u_2, u_1)=(u, u_1)-(u_2, u_1)=0$ d'où $u=u_2\in N_1$ et donc $N_2\subseteqq N_1$. Réciproquement, par symétrie des hypothèse on a $N_1 \subseteqq N_2$. Done et $N_1 = N_2$ et $N_1^\perp = N_2^\perp$ et par consèquant $M_1 = M_2$.

Soit A un opérateur fermé de domaine dense dans H. On dira que la D.K.G. (M, N) associée à A est invariante par A^* si $N\subseteqq D(A^*)$, $A^*(N)\subseteqq N$ et $A^*(M\cap D(A^*))\subseteqq M$.

Proposition 3.4. *Si* (M, N) *est une* D.K.G. *associée à A, elle est orthogonale si et seulement si elle est invariante par* A^*.

Démonstration. «Seulement si.» Soit (M, N) une D.K.G. orthogonale associée à A; d'après la démonstration du Théorème 2.4 on a $M^\perp\subseteqq D(A^*)$, $A^*(M^\perp)\subseteqq$

$\subseteq M^{\perp}$ et $A^*(N^{\perp}\cap D(A^*))\subseteq N^{\perp}$. Comme $N=M^{\perp}$ on déduit que (M, N) est invariante par A^*.

«Si.» Soit (M, N) une D.K.G. associée à A et invariante par A^*, alors $N\subseteq D(A^*)$, $A^*(N)\subseteq N$ et $A^*(M\cap D(A^*))\subseteq M$. D'autre part $A|M$ *régulier* et $A^*(M\cap D(A^*))\subseteq M$ impliquent que $A^*|M$ est *régulier*. De même $A|N$ *quasi-nilpotent* et $A^*(N)\subseteq$ $\subseteq N\Rightarrow A^*|N$ est *quasi-nilpotent*. D'où on déduit que (M, N) est une D.K.G. associée à A^*. Et d'après la démonstration du Théorème 2.4 on conclut que $(N^{\perp}, M^{\perp})$ est une D.K.G. associée à $A^{**}=A$ (car A est fermé).

Donc (M, N) et $(N^{\perp}, M^{\perp})$ sont deux D.K.G. associées à A. Par la Proposition 3.1, on en déduit que $(M, M^{\perp})$ est une D.K.G. associée à A d'où $M=N^{\perp}$.

Définition 3.5. Soit (M, N) une D.K.G. associée à A. On dit qu'elle est *non triviale* si $N\neq\{0\}$.

Théorème 3.6. *Soit A un opérateur fermé avec $D(A)$ dense dans H. Les conditions suivantes sont équivalentes:*

(i) *A admet une* D.K.G. *non triviale unique,*

(ii) $0\in\sigma(A)$ *est isolé.*

Remarque. Pour démontrer le Théorème 3.6 on a besoin des résultats suivants.

Proposition 3.7. *Soit $A\in p\Phi$ avec $D(A)$ dense dans H et soit (M, N) une* D.K.G. *non triviale associée à A. Alors les conditions suivantes sont équivalentes:*

(a) *N est unique,*

(b) $K(A)\cap N(A)=\{0\}$,

(c) $H_0(A)=N$,

(d) *$H_0(A)$ est fermé.*

Démonstration.

(a)$\Rightarrow$(b). Remarquons que $N\neq\{0\}\Rightarrow M^{\perp}\neq\{0\}$. En effet si $M^{\perp}=\{0\}$ alors $M=H$ et donc $N=\{0\}$. Soient $z\in M^{\perp}$ et $z\neq 0$. Supposons que $K(A)\cap N(A)\neq\{0\}$ et $x\in K(A)\cap N(A)$ avec $x\neq 0$. Soient $\{x_n\}_{n\geqq 0}$ et $a>0$ de la définition de $x\in K(A)$. $A\in p\Phi\Rightarrow K(A)$ est fermé, donc on peut choisir les x_n avec $n\geqq 1$ orthogonaux à $N(A|K(A))=K(A)\cap N(A)$. Les x_n étant ainsi choisis, posons:

$$N_z = \Big\{w\in H;\ w = v+\sum_{n\geqq 0}(A^n v, z)x_n \text{ avec } v\in N\Big\}.$$

N_z est bien défini. En effet, $v\in N\subseteq H_0(A)\Rightarrow v\in D(A^n)$ $\forall n\geqq 0$, d'autre part

$$\Big\|\sum_{n\geqq 0}(A^n v, z)x_n\Big\| \leqq \sum_{n\geqq 0}\|A^n v\|\,\|z\|\,\|x_n\| \leqq \|z\|\,\|u\|\sum_{n\geqq 0}a^n\,\|A^n v\|$$

et la dernière série est convergente car $v\in N\subseteq H_0(A)$. Remarquons aussi que

$N_z \subseteq D(A)$. En effet soit $w \in N_z$ alors $\exists w_j = v + \sum_{n=0}^{j} (A^n v, z) x_n$ tel que $w_j \to w$. $w_j \in D(A)$ car $N \subseteq D(A)$ et $x_n \in D(A)$ $\forall n \geqq 0$. En outre $Aw_j = Av + \sum_{n=0}^{j} (A^n v, z) Ax_n = Av + \sum_{n=1}^{j} (A^n v, z) x_{n-1}$ car $x_0 = u \in N(A)$ et $Ax_n = x_{n-1}$ par définition des x_n. Donc $Aw_j = Av + \sum_{n=0}^{j} (A^{n+1} v, z) x_n$ et le même raisonnement que plus haut montre que la suite $\{Aw_j\}$ est convergente et comme A est fermé on en déduit que $w \in D(A)$. D'autre part on vient de montrer que si $w = v + \sum_{n \geqq 0} (A^n v, z) x_n$ alors $Aw = Av + \sum_{n \geqq 0} (A^{n+1} v, z) x_n = Av + \sum_{n \geqq 0} (A^n (Av), z) x_n$. Ceci montre que $A(N_z) \subseteq N_z$.

Montrons maintenant que $N_z \subseteq H_0(A)$. Soit $w \in N_z$ on a $A^k w = A^k v + \sum_{n \geqq 0} (A^n (A^k v), z) x_n$. Par ailleurs $z \in M^\perp \subseteq H_0(A^*)$ (voir la démonstration du Théorème 2.4), donc $A^k w = A^k v + \sum_{n \geqq 0} (A^k v, (A^*)^n z) x_n$ ce qui implique

$$\|A^k w\| \leqq \|A^k v\| + \sum_{n \geqq 0} \|A^k v\| \, \|(A^*)^n z\| \, \|x_n\| \leqq \|A^k v\| + \|A^k v\| \, \|u\| \sum_{n \geqq 0} a^n \|(A^*)^n z\|.$$

$z \in H_0(A^*) \Rightarrow \sum_{n \geqq 0} a^n \|(A^*)^n z\| < \infty$, donc $\exists c > 0$ tel que $\|A^k w\| \leqq \|A^k v\| (1+c)$. D'où $\|A^k w\|^{1/k} \leqq \|A^k v\|^{1/k} (1+c)^{1/k} \to 0$ quand $k \to \infty$ car $v \in N \subseteq H_0(A)$. Donc $w \in H_0(A)$, c'est-à-dire $N_z \subseteq H_0(A)$.

Montrons que $H = M \oplus N_z$. Soit $u \in M \cap N_z$ alors $u = v + \sum_{n \geqq 0} (A^n v, z) x_n$ avec $v \in N$ ce qui implique que $u - \sum_{n \geqq 0} (A^n v, z) x_n = v \in M \cap N = \{0\} \Rightarrow v = 0$ et donc $u = 0$. D'autre part $H = M \oplus N \Rightarrow \forall u \in H$, $u = u_1 + u_2$ avec $u_1 \in M$ et $u_2 \in N$. Donc on peut écrire

$$u = \Big(u_1 - \sum_{n \geqq 0} (A^n u_2, z) x_n\Big) + \Big(u_2 + \sum_{n \geqq 0} (A^n u_2, z) x_n\Big)$$

d'où $H = M + N_z$ et donc $H = M \oplus N_z$. Par conséquent on a montré que (M, N_z) est une D.K.G. associée à A. Comme par hypothèse N est unique on en déduit que $N = N_z$. Donc pour $u \in N_z$ on a $u = v + \sum_{n \geqq 0} (A^n v, z) x_n$ ce qui implique $u - v = \sum_{n \geqq 0} (A^n v, z) x_n \in N \cap M = \{0\}$. Alors $\sum_{n \geqq 0} (A^n v, z) x_n = 0$ donc $\Big(\sum_{n \geqq 0} (A^n v, z) x_n, x\Big) = 0$ et $\sum_{n \geqq 0} (A^n v, z)(x_n, x) = 0$. Puisque les x_n pour $n \geqq 1$ sont choisis dans $[K(A) \cap N(A)]^\perp$ et que $x \in K(A) \cap N(A)$ on en déduit que $(v, z)\|x\|^2 = 0$, ce qui implique $(v, z) = 0$ et ceci $\forall v \in N$ donc $z \in N^\perp$. Or $z \in M^\perp$ donc $z = 0$ (car $H = M \oplus N \Rightarrow H = M^\perp \oplus N^\perp$). $M^\perp \neq \{0\} \Rightarrow \exists z \in M^\perp$ avec $z \neq 0$, donc $\exists N_z \neq N$ tel que (M, N_z) soit une D.K.G. associée à A, ce qui contredit l'hypothèse «N est unique». Par conséquent $x = 0$, d'où $K(A) \cap N(A) = \{0\}$.

(b)$\Rightarrow$(c). $H = M + N \Rightarrow H_0(A) = M \cap H_0(A) + N$ (car $N \subseteq H_0(A)$). Or $H_0(A) \cap M = H_0(A|M)$ et $N(A|M) = N(A) \cap M = K(A) \cap N(A)$. Donc l'hypothèse (b)$\Rightarrow$

$\Rightarrow N(A|M)=0$. D'autre part $A|M$ *régulier* $\Rightarrow R(A|M)$ est fermé. En appliquant le Théorème 1.2, on en déduit que $H_0(A|M)=\{0\}$ et donc que $H_0(A)=N$.

(c)$\Rightarrow$(d) évident car N est fermé.

(d)$\Rightarrow$(a) $\forall(M, N)$ D.K.G. associée à A, on a $H=M\oplus N$. Comme plus haut $H_0(A)=M\cap H_0(A)+N$ et $H_0(A)\cap M=H_0(A|M)$; $H_0(A)$ est fermé par hypothèse et comme M est fermé, on en dèduit que $H_0(A|M)$ est fermé comme intersection de deux fermés. D'après le Théorème 1.2 ($A|M$ étant *régulier*), on en déduit que $H_0(A|M)=\{0\}$ et donc $H_0(A)=N$. Ceci est vrai $\forall(M, N)$ D.K.G. associée à A, d'où l'unicité de N.

Proposition 3.8. *Soit* $A\in p\Phi$ *avec* $D(A)$ *dense dans* H *et soit* (M, N) *une* D.K.G. *non triviale associée à* A. *Alors les trois conditions suivantes sont équivalentes:*

(a) M *est unique,*

(b) $K(A^*)\cap N(A^*)=\{0\}$,

(c) $H_0(A^*)=M^{\perp}$,

(d) $H_0(A^*)$ *est fermé.*

Démonstration. (M, N) est une D.K.G. associée à A implique que $(N^{\perp}, M^{\perp})$ est une D.K.G. associée à A^* (voir Théorème 2.4). D'autre part $N\neq\{0\}\Rightarrow$ $\Rightarrow M^{\perp}\neq\{0\}$. Donc $(N^{\perp}, M^{\perp})$ est une D.K.G. non triviale associée à A^*. Par un raisonnement analogue à celui de la Proposition 3.7 on en déduit l'équivalence entre (a), (b), (c) et (d).

Démonstration (du Théorème 3.6).

(i)$\Rightarrow$(ii). Soit (M, N) la D.K.G. non triviale associée à A, par hypothèse unique. Les Propositions (3.7) et (3.8) impliquent que $N=H_0(A)$ et $M^{\perp}=H_0(A^*)$. D'après la Proposition 2.5 on a $H_0(A^*)^{\perp}=K(A)$ donc $M^{\perp}=K(A)^{\perp}$ c'est-à-dire $M=K(A)$. D'où finalement $H=K(A)\oplus H_0(A)$ et $H_0(A)\neq\{0\}$. En utilisant le Théorème 1.3 on en déduit (ii).

(ii)$\Rightarrow$(i). $0\in\sigma(A)$ est isolé, implique $H=K(A)\oplus H_0(A)$ et que $H_0(A)\neq\{0\}$ (voir Théorème 1.3). $(K(A), H_0(A))$ est une D.K.G. non triviale associée à A, montrons qu'elle est unique. Supposons qu'il existe une autre D.K.G. associée à A. $H_0(A)$ fermé$\Rightarrow H_0(A)=N$ (voir Proposition 3.7). Montrons maintenant que $M=K(A)$. L'inclusion «$\supseteq$» étant toujours vérifiée, il reste à montrer l'inclusion inverse. Soit $u\in M$, alors $u=u_1+u_2$ avec $u_1\in K(A)$ et $u_2\in H_0(A)$, d'où $u-u_1=$ $=u_2\in M\cap H_0(A)=M\cap N=\{0\}$ donc $u-u_1=0$. Ce qui implique que $u=u_1\in K(A)$ et que $M\subseteq K(A)$. Donc $M=K(A)$ et $N=H_0(A)$ et (i) est démontré.

Corollaire 3.9. *Soit* A *un opérateur fermé avec* $D(A)$ *dense dans* H. *Les conditions suivantes sont équivalentes:*

(1) A *admet une* D.K.G. *non triviale unique,*

(2) $0\in\sigma(A)$ *est isolé,*

(3) $H=K(A)\oplus H_0(A)$ *et* $H_0(A)\neq\{0\}$,

(4) $A\in p\Phi$ *et* $K(A)\cap N(A)=\{0\}$ *et* $K(A^*)\cap N(A^*)=\{0\}$,

(5) $A\in p\Phi$ *et* $H_0(A)$, $H_0(A^*)$ *sont fermés*,

(6) A^* *admet une* D.K.G. *non triviale unique.*

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Some aspects of nonstationarity. I

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I. Introduction

Recently, several contractive completion problems were considered in papers as [5], [6], [10], and in [5] an approach based on a lifting theorem for the representations of the algebra of upper triangular matrices was proposed.

The aim of the present paper is to point out another variant — a "nonstationary" one — for the lifting theorem of Sarason—Sz.-Nagy—Foiaş which can be also used for the above mentioned completion problems. Parametrizations with choice parameters and linear fractional maps are obtained also in this case.

The content of the paper is the following: in Section 2 we obtain a time-variant analog for some other basic results in Sz.-Nagy—Foiaş theory of contractions, as model for discrete time, time-variant linear systems. In Section 3 we describe the nonstationary variant of the lifting theorem and in the last section we show how some completion problems fit in our approach.

II. The marking model

In this section we are concerned with time-variant linear systems in the following state-space representation:

$$\begin{cases} x_{n+1} = T_n^* x_n + D_{T_n} u_n \\ y_n = D_{T_n^*} x_n - T_n u_n \end{cases} \quad n \in \mathbf{Z} \tag{1.1}$$

where $\{\mathscr{H}_n\}_{n\in\mathbf{Z}}$ is a given family of Hilbert spaces, $T_n \in \mathscr{L}(\mathscr{H}_{n+1}, \mathscr{H}_n)$ are contractions, $u_n \in \mathscr{D}_{T_n}$, $y_n \in \mathscr{D}_{T_n^*}$, $x_n \in \mathscr{H}_n$, and for a contraction $T \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ we use the standard notations $D_T = (I - T^*T)^{1/2}$ and $\mathscr{D}_T = \overline{D_T \mathscr{H}}$. Now, we consider the positive-definite kernel $\mathscr{S}$ associated by the algorithm in [7], Theorem 2.4, to the

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parameters $G_{i,i+1}=T_i$, $i\in\mathbf{Z}$, and zero in rest. Using Theorem 3.2 in [7], the Kolmogorov decomposition of $\mathscr{S}$ is simply given by the unitary operators

$$(1.2)\quad \begin{cases} W_n\colon \mathscr{K}_{n+1}\to\mathscr{K}_n \\ W_n(\dots,d_{*n-1},d_{*,n},\boxed{h_{n+1}},d_{n+1},d_{n+2},\dots)= \\ =(\dots d_{*,n-2},d_{*,n-1},\boxed{D_{T_n^*}d_{*,n}+T_nh_{n+1}},-T_n^*d_{*,n}+D_{T_n}h_{n+1},d_{n+1},d_{n+2},\dots) \end{cases}$$

where

$$\mathscr{K}_n=\dots\oplus\mathscr{D}_{T^*_{n-2}}\oplus\mathscr{D}_{T^*_{n-1}}\oplus\boxed{\mathscr{H}_n}\oplus\mathscr{D}_{T_n}\oplus\mathscr{D}_{T_{n+1}}\oplus\dots.$$

Let us pursue by introducing the main elements of the geometrical model of (1.1). Define the spaces

$$\mathscr{K}_n^+=\mathscr{H}_n\oplus\mathscr{D}_{T_n}\oplus\mathscr{D}_{T_{n+1}}\oplus\dots$$

and the isometries

$$(1.3)\qquad W_n^+\colon \mathscr{K}_{n+1}^+\to\mathscr{K}_n^+,\quad W_n^+=W_n|\mathscr{K}_{n+1}^+$$

and use the Wold decomposition for the family $\{W_k^+\}_{k\geq n}$. We denote $\mathscr{L}_n^+=\mathscr{K}_n^+\ominus\ominus W_n^+\mathscr{K}_{n+1}^+$ and according to the form (1.2) of the Kolmogorov decomposition, we identify $\mathscr{L}_n^+=W_n(\dots\oplus0\oplus\mathscr{D}_{T_n^*}\oplus\boxed{0_{\mathscr{H}_{n+1}}}\oplus0\oplus\dots)$.

Moreover, define $\mathscr{R}_n^+=\bigcap_{p=0}^{\infty}W_n\dots W_{n+p}\mathscr{K}_{n+p+1}^+$ and then

$$(1.4)\qquad \mathscr{K}_n^+=(\mathscr{L}_n^+\oplus\bigoplus_{p=1}^{\infty}W_n^+\dots W_{n+p-1}^+\mathscr{L}_{n+p}^+)\oplus\mathscr{R}_n^+.$$

Similar considerations take place for the spaces

$$\mathscr{K}_n^-=\dots\oplus\mathscr{D}_{T^*_{n-1}}\oplus\mathscr{H}_n$$

and the isometries

$$(1.5)\qquad W_n^-\colon \mathscr{K}_n^-\to\mathscr{K}_{n+1}^-,\quad W_n^-=W_n^*|\mathscr{K}_n^-.$$

We use the notation $\mathscr{L}_n^-=\mathscr{K}_n^-\ominus W_{n-1}^-\mathscr{K}_{n-1}^-$ and taking (1.2) into account, we identify $\mathscr{L}_n^-=W_n^*(\dots0\oplus0_{\mathscr{H}_n}\oplus\boxed{\mathscr{D}_{T_n}}\oplus0\oplus\dots)$. It is also useful to denote the space $\dots\oplus0\oplus\mathscr{D}_{T_n^*}\oplus\boxed{0}\oplus0\oplus\dots\subset\mathscr{K}_{n+1}$ by $\mathscr{D}_{T_n^*}^{(-1)}$ and $\dots0\oplus\boxed{0}\oplus\mathscr{D}_{T_n}\oplus0\oplus\dots\subset\mathscr{K}_n$ by $\mathscr{D}_{T_n}^{(1)}$. Another application of the Wold decomposition for the family $\{W_k^-\}_{k\leq n-1}$ will produce a decomposition

$$\mathscr{K}_n^-=(\mathscr{L}_n^-\oplus\bigoplus_{p=1}^{\infty}W_{n-1}^-\dots W_{n-p}^-\mathscr{L}_{n-p}^-)\oplus\mathscr{R}_n^-.$$

Now define the spaces

$$\mathscr{K}_n^{\text{out}}=\bigoplus_{q=1}^{\infty}W_{n-1}^*\dots W_{n-q}^*\mathscr{D}_{T^*_{n-q-1}}^{(-1)}\oplus\mathscr{D}_{T^*_{n-1}}^{(-1)}\oplus\bigoplus_{p=0}^{\infty}W_n\dots W_{n+p}\mathscr{D}_{T^*_{n+p}}^{(-1)},$$

and

$$\mathcal{K}_n^{\mathrm{inp}} = \bigoplus_{p=1}^{\infty} W_{n-1}^* \dots W_{n-p}^* \mathcal{D}_{T_{n-p}}^{(1)} \oplus \mathcal{D}_{T_n}^{(1)} \oplus \bigoplus_{q=0}^{\infty} W_n \dots W_{n+q} \mathcal{D}_{T_{n+q+1}}^{(1)}.$$

A usual condition in Sz.-Nagy—Foiaş theory (and also in system theory) is to ask $\mathcal{K}_n = \mathcal{K}_n^{\mathrm{inp}} \vee \mathcal{K}_n^{\mathrm{out}}$ for every $n \in \mathbf{Z}$. We have by direct computation using (1.2) that

$$\mathcal{K}_n \ominus (\mathcal{K}_n^{\mathrm{inp}} \vee \mathcal{K}_n^{\mathrm{out}}) =$$

$$= \{h \in \mathcal{H}_n \mid \dots \|T_{n-2}T_{n-1}h\| = \|T_{n-1}h\| = \|h\| = \|T_n^* h\| = \|T_{n+1}^* T_n^* h\| = \dots\}$$

which corresponds to Theorem I. 3.2 in [16].

Finally, we define the family of characteristic operators of the system (1.1) by the formula

$$(1.6) \qquad Q_n \colon \mathcal{K}_n^{\mathrm{inp}} \to \mathcal{K}_n^{\mathrm{out}}, \quad Q_n = P_{\mathcal{K}_n^{\mathrm{out}}}^{\mathcal{K}_n} | \mathcal{K}_n^{\mathrm{inp}}.$$

We obtain a first result concerning the geometry of the spaces $\mathcal{K}_n$.

2.1. Theorem. *For a system* (1.1) *satisfying the condition* $\mathcal{K}_n = \mathcal{K}_n^{\mathrm{inp}} \vee \mathcal{K}_n^{\mathrm{out}}$ *for every* $n \in \mathbf{Z}$, *the following relations hold:*

$$\mathcal{K}_n = \mathcal{K}_n^{\mathrm{out}} \oplus \mathcal{R}_n^+,$$

$$\mathcal{H}_n = \mathcal{K}_n^+ \ominus \{Q_n u \oplus (I - Q_n) u \mid u \in \bigoplus_{p=1}^{\infty} W_n \dots W_{n+p-1} \mathcal{D}_{T_{n+p}}^{(1)}\}.$$

Proof. The first relation is obvious. For the second one, we remark that

$$\mathcal{K}_n^+ = \mathcal{H}_n \vee \bigvee_{p=1}^{\infty} W_n \dots W_{n+p-1} \mathcal{H}_{n+p}$$

and, as

$$W_n \mathcal{D}_{T_n^*}^{(-1)} \oplus W_n \mathcal{K}_{n+1} = \mathcal{H}_n \oplus \mathcal{D}_{T_n}^{(1)},$$

one obtains

$$\mathcal{K}_n^+ \subset \mathcal{H}_n \oplus \mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \dots .$$

The converse inclusion is clear, consequently

$$\mathcal{H}_n = \mathcal{K}_n^+ \ominus (\mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \dots),$$

which completes the proof.

Then we introduce the marking model. The marking operators appear as the main elements involved by the Kolmogorov decomposition of an arbitrary positive-definite kernel. In our case, define the spaces:

$$\mathcal{M}_+ = \bigoplus_{n \in \mathbf{Z}} \mathcal{D}_{T_n^*} \quad \text{and} \quad \mathcal{M}_- = \bigoplus_{n \in \mathbf{Z}} \mathcal{D}_{T_n}$$

and the operators

$$(1.7)\qquad \begin{cases} M_n^+: \bigoplus_{k\geq n+1} \mathcal{D}_{T_k^*} \to \bigoplus_{k\geq n} \mathcal{D}_{T_k^*}, \\ M_n^+(d_{*,n+1}, d_{*,n+2}, \ldots) = (0, d_{*,n+1}, d_{*,n+2}, \ldots) \end{cases}$$

$$(1.8)\qquad \begin{cases} M_n^-: \bigoplus_{k\geq n+1} \mathcal{D}_{T_k} \to \bigoplus_{k\geq n} \mathcal{D}_{T_k}, \\ M_n^-(d_{n+1}, d_{n+2}, \ldots) = (0, d_{n+1}, d_{n+2}, \ldots). \end{cases}$$

Our goal is to obtain identifications for $\mathcal{K}_n^{\mathrm{inp}}$, $\mathcal{K}_n^{\mathrm{out}}$, $\mathcal{K}_n$ and the characteristic operators in terms of the marking operators (1.7) and (1.8) and the marking spaces $\mathcal{M}_+$ and $\mathcal{M}_-$. For this aim, we introduce the following unitary operators

$$(1.9)\qquad \begin{cases} \Phi_n^+: \mathcal{K}_n^{\mathrm{out}} \to \mathcal{M}_+, \\ \Phi_n^+(\ldots \oplus W_{n+1}^* d_{*,n-2}^{(-1)} \oplus d_{*,n-1}^{(-1)} \oplus W_n d_{*,n}^{(-1)} \oplus \ldots) = (\ldots, d_{*,-1}, \boxed{d_{*,0}}, d_{*,1}, \ldots), \end{cases}$$

$$(1.10)\qquad \begin{cases} \Phi_n^-: \mathcal{K}_n^{\mathrm{inp}} \to \mathcal{M}_-, \\ \Phi_n^-(\ldots \oplus W_{n-1}^* d_{n-1}^{(1)} \oplus d_n^{(1)} \oplus W_n d_{n+1}^{(1)} \oplus \ldots) = (\ldots, d_{-1}, \boxed{d_0}, d_1, \ldots), \end{cases}$$

where

$$d_{*,n}^{(-1)} = (\ldots, 0, d_{*,n}, \boxed{0}, 0, \ldots) \in \mathcal{K}_{n+1}, \quad d_{*,n} \in \mathcal{D}_{T_n^*}$$

and

$$d_n^{(1)} = (\ldots, 0, \boxed{0}, d_n, 0, \ldots) \in \mathcal{K}_n, \quad d_n \in \mathcal{D}_{T_n}.$$

The first remark is that for every $n\in\mathbf{Z}$ we get

$$\Phi_n^+ Q_n (\Phi_n^-)^{-1} = \Theta$$

where Θ is the transfer operator of the system (1.1) — see [11] for definition. Θ is a lower triangular operator such that its matricial elements are $\Theta_{ij} = D_{T_j} T_{j+1} \ldots T_{i-1} D_{T_i^*}$ for $i\in\mathbf{Z}$, $j<i$ and $\Theta_{ii} = -T_i^*$, $i\in\mathbf{Z}$. Θ is a contraction and we obtain the following identification of $\mathcal{K}_n$ in the model given by the marking operators: first we define the unitary operator

$$(1.11)\qquad \Phi_{\mathcal{R}_n^+}: \mathcal{R}_n^+ \to \overline{D_\Theta \mathcal{M}_-}, \quad \Phi_{\mathcal{R}_n^+}(I-Q_n)k = D_\Theta \Phi_n^- k, \quad k\in\mathcal{K}_n^{\mathrm{inp}}$$

then

$$(1.12)\qquad \Psi_n: \mathcal{K}_n \to \mathcal{M}_+ \oplus \overline{D_\Theta \mathcal{M}_-}, \quad \Psi_n = \Phi_n^+ \oplus \Phi_{\mathcal{R}_n^+}$$

and Ψ_n is a unitary operator yielding a natural identification of $\mathcal{K}_n$ in the marking model. Moreover, we have the following result which constitutes the time-variant analogue of the Sz.-Nagy—Foiaş functional model of a contraction.

Theorem 2.2. *For a system* (1.1) *satisfying the condition* $\mathscr{K}_n=\mathscr{K}_n^{\text{inp}}\vee\mathscr{K}_n^{\text{out}}$ *for every* $n\in\mathbf{Z}$, *the following relations hold through the identifications* Ψ_n:

$$\mathscr{H}_n=\Big(\bigoplus_{k\geq n}\mathscr{D}_{T_k^*}\oplus\overline{D_\Theta\mathscr{M}_-}\Big)\ominus\Big\{\Theta v\oplus D_\Theta v\,|\,v\in\bigoplus_{k\geq n}\mathscr{D}_{T_k}\Big\},$$

$$T_n(u_+\oplus v_-)=P_{\mathscr{H}_n}(M_n^+u_+\oplus M_n^-v_-),\quad u_+\oplus v_-\in\mathscr{H}_n.$$

Proof. From (1.4), (1.12) and (1.11) it follows that

$$\Psi_n\mathscr{K}_n^+=\Phi_n^+\Big(\mathscr{L}_n^+\oplus\bigoplus_{p=1}^{\infty}W_n^+\dots W_{n+p-1}^+\mathscr{L}_{n+p}^+\Big)\oplus\overline{D_\Theta\mathscr{M}_-}$$

and by (1.9) we have

$$\Phi_n^+\Big(\mathscr{L}_n^+\oplus\bigoplus_{p=1}^{\infty}W_n^+\dots W_{n+p-1}^+\mathscr{L}_{n+p}^+\Big)=\bigoplus_{k\geq n}\mathscr{D}_{T_n^*}.$$

In a similar way,

$$\mathscr{D}_{T_n}^{(1)}\oplus W_n\mathscr{D}_{T_{n+1}}^{(1)}\oplus\dots=\bigoplus_{k\geq n}\mathscr{D}_{T_k}$$

and the first relation follows from Theorem 2.1. For the second relation we have to use the more remark that

$$\Phi_n^+W_n(\Phi_{n+1}^+)^*\ \Big|\ \bigoplus_{k\geq n+1}\mathscr{D}_{T_k^*}=M_n^+$$

$$\Phi_n^-W_n(\Phi_{n+1}^-)^*\ \Big|\ \bigoplus_{k\geq n+1}\mathscr{D}_{T_k}=M_n^-.$$

Remark 2.3. The inverse problem of realization a given lower triangular contraction as a transfer operator of a certain system is treated for instance in [11] and [2].

III. Nonstationary lifting

In this section we describe a nonstationary variant for the lifting theorem of Sarason—Sz.-Nagy—Foiaş.

This variant is inspired by similar phenomena in the study of nonstationary processes (see [12], [13], [7]) and the difference from the "stationary" variant of Sarason—Sz.-Nagy—Foiaş is not structural, but only one of complexity. Consequently, we will have only to indicate the necessary changes, the proofs following the known ones. Fix two integers $-\infty\leq M<\infty$, $-\infty<N\leq\infty$, $M\leq N$ and two families $\{T_n\}_{M\leq n\leq N}$, $\{T_n'\}_{M\leq n\leq N}$ of contractions (the extremal indices are attained only for finite M and N), $T_n\in\mathscr{L}(\mathscr{H}_{n+1},\mathscr{H}_n)$, $T_n'\in\mathscr{L}(\mathscr{H}_{n+1}',\mathscr{H}_n')$. Let $\{A_n\}_{M\leq n\leq N+1}$ be a family of contractions, $A_n\in\mathscr{L}(\mathscr{H}_n,\mathscr{H}_n')$ and suppose that it intertwines $\{T_n\}$ and $\{T_n'\}$, i.e.

$$T_n'A_{n+1}=A_nT_n$$

for $M \leqq n \leqq N$. For $\{T_n\}_{M \leqq n \leqq N}$ consider its associated kernel by the rule mentioned at the beginning of Section 2 and let $\{W_n\}_{M \leqq n \leqq N}$, $W_n \in \mathscr{L}(\mathscr{K}_{n+1}, \mathscr{K}_n)$ be its Kolmogorov decomposition, always written in the form (1.2). We have similar objects associated to $\{T'_n\}_{M \leqq n \leqq N}$. Now, the following result extends the lifting theorem of Sarason—Sz.-Nagy—Foiaş. Denote by P_n the orthogonal projection of $\mathscr{K}_n^+$ onto $\mathscr{H}_n$ and similarly, P'_n.

Theorem 3.1. *The set*

$$\mathrm{CID}\left(\{A_n\}_{M \leqq n \leqq N+1}\right) =$$
$$= \left\{\{B_n\}_{M \leqq n \leqq N+1} \mid B_n \text{ are contractions in } \mathscr{L}(\mathscr{K}_n^+, \mathscr{K}_n'^+),\ W_n'^+ B_{n+1} = B_n W_n^+,\ P'_n B_n = A_n P_n\right\}$$

is nonvoid.

Proof. Let $X_{ij}^{(n)}$ be the matrix of B_n, then writing the intertwining conditions, one gets:

$$X_{11}^{(n)} = A_n, \quad X_{1j}^{(n)} = 0, \quad j > 1$$
$$X_{ij}^{(n)} = 0, \quad j > i$$
$$X_{21}^{(n)} T_n + X_{22}^{(n)} D_{T_n} = D_{T'_n} A_{n+1}$$
$$X_{k1}^{(n)} T_n + X_{k2}^{(n)} D_{T_n} = X_{k-1,1}^{(n+1)}, \quad k \geqq 3$$

and

$$X_{ij}^{(n)} = X_{i-1,j-1}^{(n+1)}, \quad i, j \geqq 3.$$

Define the operators

$$S_{k-1,n}\colon \mathscr{D}_{T_n^*} \to \mathscr{D}_{T'_n} \quad S_{k-1,n} = X_{k1}^{(n)} D_{T_n^*} - X_{k2}^{(n)} T_n^* \tag{3.1}$$

such that the finite sections of B_n are contractions if and only if the operators

$$C_{kn} = \begin{bmatrix} A_n T_n \dots T_{n+k-1}, & \dots, & A_n T_n D_{T_{n+1}^*}, & A_n D_{T_n^*} \\ & \dots, & & S_{1n} \\ & \dots, & S_{1,n+1}, & S_{2,n+1} \\ & & \ddots & \\ D_{T'_{n+k-1}} A_{n+k}, & S_{1,n+k-1}, & \dots, & S_{k-1,n+k-1} \end{bmatrix} \tag{3.2}$$

are also contractions for $k \geqq 1$.

If we define $C_{0n} = A_n T_n = T'_n A_{n+1}$, then there exist contractions $Y_n\colon \mathscr{D}_{C_{0n}} \to \mathscr{D}_{T'_n}$ and $\tilde{Y}_n\colon \mathscr{D}_{T_n^*} \to \mathscr{D}_{C_{0n}^*}$ such that $A_n D_{T_n^*} = D_{C_{0n}^*} \tilde{Y}_n$ and $D_{T'_n} A_{n+1} = Y_n D_{C_{0n}}$ and using [4], [9] there exists an operator S_{1n} such that C_{1n} is a contraction. Now, the same approximation procedure as in [3] finishes the proof.

We can continue the analysis of the set $\mathrm{CID}\left(\{A_n\}_{M \leqq n \leqq N+1}\right)$ in order to derive results similar to those in [1], [3]. That is, a parametrization with a family of free parameters (generalizing the choice sequences in [3]) and a parametrization with lower triangular contractions — a Schur type formula — are obtained.

For a sequence of contractions $\{G_1, G_2, \dots\}$, $G_1 \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$, $G_k \in \mathscr{L}(\mathscr{D}_{G_{k-1}}, \mathscr{H}')$, $L(G_1, G_2, \dots)$ is the row contraction determined by these parameters,

$$L(G_1, G_2, \dots) = (G_1, D_{G_1^*} G_2, \dots, D_{G_1^*} \dots D_{G_{k-1}^*} G_k, \dots)$$

and similar considerations hold for column contractions, denoted by $C(G_1, G_2, \dots)$.

Theorem 3.2. *There exists a one-to-one correspondence between*

$$\mathrm{CID}\,(\{A_n\}_{M \leqq n \leqq N+1})$$

and the families of contractions $\{G_{ij}\}$ *such that* $G_{1n} \in \mathscr{L}(\mathscr{D}_{G_{Y_n}}, \mathscr{D}_{G^*_{\tilde{Y}_n}})$, $M \leqq n \leqq N$ *and* $G_{ij} \in \mathscr{L}(\mathscr{D}_{G_{i-1,j}}, \mathscr{D}_{G^*_{i-1,j+1}})$ *for* $i \geqq 2$, $M \leqq j \leqq N$. *The correspondence is explicitely taken by the formula:*

$$S_{kn} = L(Y_n, G_{1n}, G_{2n}, \dots, G_{k-1,n}) Q_{k-1,n} C(\tilde{Y}_{n-k+1}, G_{1,n-k+1}, G_{2,n-k+2}, \dots, G_{k-1,n-1}) +$$
$$+ D_{Y_n^*} D_{G_{1n}^*} \dots D_{G_{k-1,n}^*} G_{kn} D_{G_{k-1,n-1}} \dots D_{G_{1,n-k+1}} D_{\tilde{Y}_{n-k+1}},$$

where the operators Q_{kn} *can be also described in terms of the parameters* G_{ij}.

Proof. We only skech the beginning, the rest paralleling the proof in [8] of a slight modified variant of the main algorithm in [3].

First of all, $Q_{0n} = -C_{0n}^*$ for $M \leqq n \leqq N$. Denote by F_{kn} the $k \times k$ principal submatrix of $C_{k+1,n}$ and by direct computations, we have

$$C_{2n} = \begin{bmatrix} F_{1n} & D_{F_{1n}^*} \tilde{\Omega}_{1n}^* C(Y_n, G_{1n}) \\ L(\tilde{Y}_{n+1}, G_{1,n+1}) \Omega_{1n} D_{F_{1n}} & S_{2,n+1} \end{bmatrix}$$

where Ω_{1n} and $\tilde{\Omega}_{1n}$ are obvious identifications of the defects of F_{1n}. Using once again [4], [9] we get the desired formula for S_{2n} with

$$Q_{1n} = -\Omega_{1n} F_{1n}^* \tilde{\Omega}_{1n}^*.$$

Then we compute

$$Q_{1n} \begin{bmatrix} D_{C_{0n}^*} & -C_{0n} \tilde{Y}_n^* \\ 0 & D_{\tilde{Y}_n^*} \end{bmatrix} = -\Omega_{1n} F_{1n}^* \tilde{\Omega}_{1n}^* \begin{bmatrix} D_{C_{0n}^*} & -C_{0n} \tilde{Y}_n^* \\ 0 & D_{\tilde{Y}_n^*} \end{bmatrix} =$$

$$= -\Omega_{1n} F_{1n}^* D_{F_{1n}^*} = -\Omega_{1n} D_{F_{1n}} F_{1n}^* = \begin{bmatrix} D_{C_{0,n+1}} & -C_{0,n+1}^* Y_{n+1} \\ 0 & D_{Y_{n+1}} \end{bmatrix} F_{1n}^*.$$

As in [8] we find

$$Q_{1n} = -\begin{bmatrix} C_{0n}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{Y_{n+1}^*} a_n & D_{Y_{n+1}^*} b_n \\ Y_{n+1}^* a_n & Y_{n+1}^* b_n \end{bmatrix}$$

with $a_n a_n^* + b_n b_n^* = I$ and we define the operator

$$V_{0n}: \mathscr{D}_{C_{0n}^*} \oplus \mathscr{D}_{Y_n^*} \to \mathscr{D}_{C_{0n}} \oplus \mathscr{D}_{Y_{n+1}}$$

$$V_{0n} = \begin{bmatrix} D_{Y_{n+1}^*} a_n & D_{Y_{n+1}^*} b_n \\ -Y_{n+1}^* a_n & -Y_{n+1}^* b_n \end{bmatrix}.$$

From now on we can continue as in [8].

Remark 3.3. Using Theorem 5.2 in [2] and Theorem 3.2 above, a parametrization of $\mathrm{CID}\left(\{A_n\}_{M \leqq n \leqq N+1}\right)$ with lower triangular contractions can be derived, together with corresponding Schur type formulae as in Corollary 6.1 in [3].

IV. Applications

In this section we show the way some completion problems can be solved using Theorem 3.1.

(A) For fixed operators $\left(C_{j+r,j} \in \mathscr{L}(\mathscr{H}_j, \mathscr{H}'_{j+r}) \mid j \geqq 0,\ 0 \leqq r \leqq N\right)$ we find conditions for the existence of lower triangular contractive extensions. This problem can be viewed as a "nonstationary" Carathéodory—Fejér problem and can be solved as in [15].

We define for $n \geqq 0$

$$T_n: \bigoplus_{k=1}^{N} \mathscr{H}'_{n+k} \to \bigoplus_{k=0}^{N-1} \mathscr{H}'_{n+k}$$

$$T_n = \begin{bmatrix} 0 & 0 \dots & 0 \\ I & 0 \dots & 0 \\ & \dots & \\ 0 & 0 \quad I & 0 \end{bmatrix}$$

and

$$A_n: \bigoplus_{k=0}^{N-1} \mathscr{H}_{n+k} \to \bigoplus_{k=0}^{N-1} \mathscr{H}'_{n+k}$$

$$A_n = \begin{bmatrix} C_{nn} & 0 & 0 \dots & 0 \\ C_{n+1,n} & C_{n+1,n+1} & 0 \dots & 0 \\ & \dots & & \\ C_{n+N-1,n} & \dots & & C_{n+N-1,n+N-1} \end{bmatrix}.$$

We have that $T_n A_{n+1} = A_n T_n$ and if we suppose that A_n are contractions, we can use Theorem 3.1 in order to show that there exists a family of contractions

$\{B_n\}_{n\geqq 0}$ such that

$$M_n B_{n+1} = B_n M_n, \quad P_n B_n = A_n P_n$$

where M_n are marking operators as those given by (1.7) or (1.8).

Using an adaptation of Lemma V. 3.2 in [16], $\{B_n\}_{n\geqq 0}$ gives rise to a contractive lower triangular extension of the given family of operators.

Proposition 4.1. *In order that the family* $(C_{j+r,j} \mid j\geqq 0,\ 0\leqq r\leqq N)$ *has a contractive lower triangular extension it is necessary and sufficient that* A_n *are contractions for* $n\geqq 0$.

Moreover, Theorem 3.1 and Remark 3.3 give parametrizations for all the solutions.

(B) Theorem 3.1 can be used to solve completion problems with a finite number of data, those named as Nehari completions in [5]. We indicate here (for simplicity) only the very particular case of completing

$$\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & \end{bmatrix}$$

to a contraction. Take

$$A_0 = (C_{00}, C_{01}), \quad A_1 = \begin{bmatrix} C_{00} \\ C_{10} \end{bmatrix}, \quad T_0' = (I, 0), \quad T_0 = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

then $T_0' A_1 = A_0 T_0$. Moreover,

$$W_0'^{+} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad W_0^{+} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

and if A_0 and A_1 are supposed to be contractions, then Theorem 3.1 asserts the existence of a contraction $\begin{bmatrix} C_{00} & C_{01} \\ C_{21} & C_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C_{00} \\ C_{10} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Consequently, $C_{21} = C_{10}$ and a contractive completion of the given (C_{00}, C_{01}, C_{10}) is

$$\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{22} \end{bmatrix}.$$

This shows that Theorem 3.1 (together with the parametrization in Theorem 3.2) for $M=N=0$ is equivalent with [4] and [9].

(C) Another application here is an extension of Theorem 5 in [5] and of a similar result in [14].

Proposition 4.2. *Let A and B be two lower triangular operators, $A\in\mathscr{L}(\bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n, \bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n'')$, $B\in\mathscr{L}(\bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n, \bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n')$. Then a necessary and sufficient condition for the existence of a lower triangular contraction $C\in\mathscr{L}(\bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n', \bigoplus_{n\in\mathbf{Z}}\mathscr{H}_n'')$ such that $A=CB$ is that $A^*A\leqq B^*B$.*

Proof. Take $A_n=A\mid\bigoplus_{k\geqq n}\mathscr{H}_k$, $B_n=B\mid\bigoplus_{k\geqq n}\mathscr{H}_k$ and M_n, M_n' and M_n'' be marking operators as (1.7) and (1.8) such that

$$M_n'A_{n+1} = A_nM_n, \quad M_n''B_{n+1} = B_nM_n.$$

Since A and B are lower triangular, then $A_n^*A_n\leqq B_n^*B_n$ for $n\in\mathbf{Z}$ and there exist uniquely determined contractions X_n: $\overline{\operatorname{Ran} B_n}\to\overline{\operatorname{Ran} A_n}$ such that $A_n=X_nB_n$. From now on we can follow [14].

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Uniform boundedness theorems for k-triangular set functions

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In a recent paper, we have obtained a generalization of the classical boundedness Dieudonné theorem ([9], Prop. 9), in the setting of finitely additive group valued-functions ([14], (3.2)).

The purpose of this paper is to obtain an analogous result ((3.3)), in the setting of semigroup valued k-triangular functions. For this, firstly we establish that Nikodym's boundedness theorem holds for k-triangular exhaustive functions on a ring with the Subsequential Interpolation Property ((1.6)). This proposition yields some recent results of E. Pap as special cases (see [21], [23], [24]). We apply (3.3) to obtain again a Dieudonné type theorem for finitely additive group-valued functions (Corollary (3.8), see also [14], (4.2)).

1. Let X be a commutative semigroup with neutral element 0; let p be a semi-invariant pseudometric on X, namely a pseudometric satisfying the inequality

$$p(x+z, y+z) \leqq p(x, y) \quad \forall x, y, z \in X,$$

or, equivalenty, the inequality

$$p(x+x', y+y') \leqq p(x, y)+p(x', y') \quad \forall x, x', y, y' \in X.$$

Let $\mathbf{R}^+=[0, +\infty[$, $\overline{\mathbf{R}}^+=[0, +\infty]$. To p there corresponds the function

$$|\ |\colon x \in X \to p(x, 0) \in \mathbf{R}^+$$

for which

$$|0| = 0$$

$$\big||x|-|y|\big| \leqq |x+y| \leqq |x|+|y| \quad \forall x, y \in X,$$

([22], [23]).

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We will denote by $(X, | \ |)$ the uniform semigroup $(X, \mathscr{U}_p)$, where $\mathscr{U}_p$ is the uniformity of X generated by the pseudometric p ([11], [27]). We say that a subset Y of X is *bounded* if $\sup_{y \in Y} |y| < +\infty$.

Let $\mathscr{R}$ be a ring of subsets of a set S and φ a function from $\mathscr{R}$ to $(X, | \ |)$. We say that φ is *bounded* if the set $\varphi(\mathscr{R})$ is a bounded subset of X.

Let $k \in \mathbf{R}^+$. We say that φ is *k-triangular* if $\varphi(\emptyset)=0$ and for any disjoint sets A and B from $\mathscr{R}$,

$$|\varphi(A)| - k\,|\varphi(B)| \leqq |\varphi(A \cup B)| \leqq |\varphi(A)| + k\,|\varphi(B)|.$$

It is easy to see that φ is k-triangular if and only if

$$\varphi(\emptyset) = 0$$

and, for any sets C, D from $\mathscr{R}$, we have

$$\big||\varphi(C)| - |\varphi(D)|\big| \leqq k\,|\varphi(C \setminus D)| + k\,|\varphi(D \setminus C)|$$

([16]).

Moreover, a function φ k'-triangular is k-triangular for each $k \geqq k'$ and φ k-triangular for $k \in]0, 1[$ implies $|\varphi(X)|=0$ for each $X \in \mathscr{R}$. Hence below we will consider k-triangular functions with $k \geqq 1$.

Let $\mathscr{G}$ be a lattice contained in $\mathscr{R}$; we say that a function φ from $\mathscr{R}$ to $(X, | \ |)$ is *$\mathscr{G}$-exhaustive* if, for every disjoint sequence $(G_n)_{n \in N}$ in $\mathscr{G}$, we have

$$\lim_n \varphi(G_n) = 0;$$

an $\mathscr{R}$-exhaustive function is called exhaustive.

We say that a function φ from $\mathscr{R}$ to $(X, | \ |)$ is *order continuous* if for every decreasing sequence $(A_n)_{n \in N}$ in $\mathscr{R}$ such that $\bigcap_{n \in N} A_n = \emptyset$

$$\lim_n \varphi(A_n) = 0.$$

We write, for every $\mathscr{H} \subseteqq \mathscr{R}$ and $A \in \mathscr{R}$,

$$\mathscr{H}_A = \{H \in \mathscr{H}: \ H \subseteqq A\}$$

$$\mathscr{H} \cap A = \{H \cap A, \ H \in \mathscr{H}\}.$$

Let φ be a function from $\mathscr{R}$ to $(X, | \ |)$; its *semivariation* (supremitation or supremacy in [15], [16]) is the function

$$\tilde{\varphi}: A \in \mathscr{R} \to \sup_{B \in \mathscr{R}_A} |\varphi(B)| \in \overline{\mathbf{R}}^+$$

([11], [12], [19]).

We have

$$\tilde{\varphi}(\emptyset) = 0 \quad \text{if} \quad |\varphi(\emptyset)| = 0,$$

$$|\varphi(A)| \leqq \tilde{\varphi}(A) \quad \forall A \in \mathscr{R},$$

$$A \subseteq B \Rightarrow \tilde{\varphi}(A) \leqq \tilde{\varphi}(B).$$

Moreover $\tilde{\varphi}$ is k-subadditive if the function

$$A \in \mathscr{R} \to |\varphi(A)| \in \mathbf{R}^+$$

is k-subadditive [1]); $\tilde{\varphi}$ is exhaustive iff φ is exhaustive ([12], Lemma (2.2)).

Now, we give the proof of:

(1.1). *Let $\mathscr{R}$ be a ring of subsets of S and φ a k-triangular and exhaustive function from $\mathscr{R}$ to $(X, |\ |)$. Then φ (and therefore $\tilde{\varphi}$) is bounded*[2]).

Suppose the contrary. Then by Lemma (2.1) of [19] we can find $A_0 \in \mathscr{R}$ such that for every $A \in \mathscr{R}$

$$|\varphi(A \setminus A_0)| \leqq 1.$$

Therefore the set $\varphi(\mathscr{R}_{A_0})$ is not bounded[3]) and we can find $B_1 \in \mathscr{R}_{A_0}$ such that

$$|\varphi(B_1)| > k + |\varphi(A_0)|.$$

Hence we have also

$$|\varphi(A_0 \setminus B_1)| \geqq \frac{1}{k} \big||\varphi(B_1)| - |\varphi(A_0)|\big| > 1$$

and or $\varphi(\mathscr{R}_{B_1})$ or $\varphi(\mathscr{R}_{A_0 \setminus B_1})$ is not bounded.

Then we write $A_1 = B_1$ and $C_1 = A_0 \setminus B_1$ if $\varphi(\mathscr{R}_{B_1})$ is not bounded; on the contrary, we write $A_1 = A_0 \setminus B_1$ and $C_1 = B_1$.

It is clear now that we can obtain, as in [19], Theorem (2.2), a sequence $(C_n)_{n \in N}$ of mutually disjoint sets of $\mathscr{R}$ such that

$$|\varphi(C_n)| > 1 \quad \forall n \in \mathbf{N},$$

a contradiction with the assumption that φ is exhaustive.

(1.2). *Let $\mathscr{R}$ be a ring of subsets of S and let φ an order continuous function from $\mathscr{R}$ to $(X, |\ |)$. If the function*

$$A \in \mathscr{R} \to |\varphi(A)|$$

is k-subadditive, this function and the semivariation of φ, $\tilde{\varphi}$, are also countably k-subadditive[4]).

Let $(A_n)_{n\in N}$ be a disjoint sequence of elements of $\mathscr{R}$ such that $\bigcup_{n\in\mathbf{N}} A_n\in\mathscr{R}$. Then, for every $n\in N$ $(n\geqq 2)$ and for every $A\in\mathscr{R}$,

$$|\varphi(\bigcup_{n\in\mathbf{N}} A_n\cap A)| \leqq |\varphi(\bigcup_{i\leqq n} A_i\cap A)|+k|\varphi(\bigcup_{i>n} A_i\cap A)| \leqq$$

$$\leqq \varphi(A_1\cap A)+k\sum_{1<i\leqq n}|\varphi(A_i\cap A)|+k|\varphi(\bigcup_{i>n} A_i\cap A)|.$$

Taking limits in the above inequality, we obtain, for each $A\in\mathscr{R}$,

$$|\varphi(\bigcup_{n\in\mathbf{N}} A_n\cap A)| \leqq |\varphi(A_1\cap A)|+k\sum_{n\geqq 2}|\varphi(A_n\cap A)|,$$

and also

$$\tilde{\varphi}(\bigcup_{n\in\mathbf{N}} A_n) \leqq \tilde{\varphi}(A_1)+k\sum_{n\geqq 2}\tilde{\varphi}(A_n);$$

this completes the proof.

Corollary (1.2). *If φ is an order continuous k-subadditive function defined on the ring $\mathscr{R}$ with values in $\mathbf{R}^+$, then φ and its semivariation $\tilde{\varphi}$ are also countably k-subadditive.*

(1.3). *Let $\mathscr{R}$ be a quasi σ-ring of subsets of S and let $(\varphi_n)_{n\in\mathbf{N}}$ be a sequence of exhaustive functions from $\mathscr{R}$ to $(X, |\ |)$. Then, for each disjoint sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathscr{R}$, there exist a subsequence $(A_{n_r})_{r\in\mathbf{N}}$ of $(A_n)_{n\in\mathbf{N}}$ and a quasi σ-ring $\mathscr{S}$ contained in $\mathscr{R}$ such that $A_{n_r}\in\mathscr{S}$ for each $r\in\mathbf{N}$, such that for every $n\in\mathbf{N}$ the restriction of φ_n to $\mathscr{S}$ is order continuous*[5]).

Let, for each $n\in\mathbf{N}$, $\tilde{\varphi}_n$ be the semivariation of φ_n and let

$$\eta\colon A\in\mathscr{R} \to \sum_{n\in\mathbf{N}}\frac{1}{2^n}\inf\{1, \tilde{\varphi}_n(A)\}\in\mathbf{R}^+;$$

it is easy to see that η is an exhaustive function such that

$$\eta(A)\leqq\eta(B) \quad\text{if}\quad A\subseteqq B,\quad A, B\in\mathscr{R}\text{[6]}).$$

Let $(A_n)_{n\in\mathbf{N}}$ be a disjoint sequence of sets of $\mathscr{R}$; then we can find a subsequence $(A_{n_r})_{r\in\mathbf{N}}$ of $(A_n)_{n\in\mathbf{N}}$ and a quasi σ-ring $\mathscr{S}$ contained in $\mathscr{R}$ such that $A_{n_r}\in\mathscr{S}$, for each $r\in\mathbf{N}$, and the restriction of η to $\mathscr{S}$ is order continuous[7]).

Hence, if $(B_p)_{p\in\mathbf{N}}$ is a decreasing sequence of sets of $\mathscr{S}$ such that $\bigcap_{p\in\mathbf{N}} B_p=\emptyset$, we have, for each $n\in\mathbf{N}$,

$$\lim_p \varphi_n(B_p\cap B) = 0,$$

uniformly with respect to $B\in\mathscr{S}$; namely, for each $n\in\mathbf{N}$, the restriction of φ_n to $\mathscr{S}$ is order continuous.

(1.4). *Let* $\mathscr{R}$ *be a ring of subsets of* S *and let* Φ *be a set of k-triangular functions from* $\mathscr{R}$ *to* $(X, |\ |)$, *such that*

a) $\Phi(A)$ *is bounded for every* $A \in \mathscr{R}$,

b) *for every sequence* $(\varphi_n)_{n\in \mathbf{N}}$ *of elements of* Φ *and for every disjoint sequence* $(A_n)_{n\in\mathbf{N}}$ *of sets of* $\mathscr{R}$ *there exists an infinite subset* M *of* $\mathbf{N}$ *such that* $\bigcup_{n\in M} \{\varphi_n(A_n)\}$ *is bounded.*

Then $\Phi(\mathscr{R})$ *is bounded.*[8])

Suppose the contrary. Then there are two possibilities.

Case I: There exists $A\in\mathscr{R}$ such that $\Phi(\mathscr{R}_A)$ is not bounded.
In this case, firstly we prove:

c) *for every* $A\in\mathscr{R}$ *such that* $\Phi(\mathscr{R}_A)$ *is not bounded and for every* $n\in\mathbf{N}$, *there exists* $(\varphi, B)\in\Phi\times\mathscr{R}_A$ *such that*

$$|\varphi(B)| > n \quad \textit{and} \quad \Phi(\mathscr{R}_B) \textit{ is not bounded.}$$

In fact, suppose that there exist $n_0\in\mathbf{N}$ and $A_0\in\mathscr{R}$ such that $\Phi(\mathscr{R}_{A_0})$ is not bounded such that for every $(\varphi, B)\in\Phi\times\mathscr{R}_{A_0}$, $|\varphi(B)|>n_0$ implies that $\Phi(\mathscr{R}_B)$ is bounded. Let $(\bar\varphi, B)\in\Phi\times\mathscr{R}_{A_0}$ such that $|\bar\varphi(B)|>2kn_0$; therefore

$$|\bar\varphi(B)| > n_0 \quad \text{and} \quad |\bar\varphi(A_0\setminus B)| > n_0.$$

Hence, $\Phi(\mathscr{R}_B)$ and $\Phi(\mathscr{R}_{A_0\setminus B})$ being bounded, $\Phi(\mathscr{R}_{A_0})$ is bounded, a contradiction.

Let now $A_1\in\mathscr{R}$ such that $\Phi(\mathscr{R}_{A_1})$ is not bounded and $r(1)$ such that

$$|\varphi(A_1)| \leqq r(1) \quad \forall\varphi\in\Phi;$$

by c) there exists $(\varphi_1, A_2)\in\Phi\times\mathscr{R}_{A_1}$ such that

$$|\varphi_1(A_2)| > k+r(1) \quad \text{and} \quad \Phi(\mathscr{R}_{A_2}) \text{ is not bounded.}$$

Continuing by induction, we can find a decreasing sequence $(A_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{R}_n$, a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions of Φ and a sequence $\big(r(n)\big)_{n\in\mathbf{N}}$ of natural numbers such that for every $n\in\mathbf{N}$,

$$|\varphi_n(A_n)| \leqq r(n), \quad |\varphi_n(A_{n+1})| > kn+r(n), \quad \Phi(\mathscr{R}_{A_n}) \text{ are not bounded.}$$

Finally, if we write $C_n=A_n\setminus A_{n+1}$ for each $n\in\mathbf{N}$, $(C_n)_{\in\mathbf{N}}$ is a disjoint sequence of sets of $\mathscr{R}$ such that

$$|\varphi_n(C_n)| > n \quad \forall\, n\in\mathbf{N},$$

a contradiction with b).

Case II: For every $A\in\mathscr{R}$ the set $\Phi(\mathscr{R}_A)$ is bounded.

In this case, if we denote by $\mathscr{R}'_A$ the ring of sets of $\mathscr{R}$ disjoint from A, we have that $\Phi(\mathscr{R}'_A)$ is not bounded, for each $A\in\mathscr{R}$. Then, we put $A_0=\emptyset$ and we choose $\varphi_2\in\Phi$ and $A_1\in\mathscr{R}$ such that $|\varphi_1(A_1)|\geqq 1$. Continuing by induction, we find for every $n\in\mathbf{N}$, $\varphi_n\in\Phi$ and $A_n\in\mathscr{R}'_{\bigcup_{1\leqq i\leqq n-1} A_i}$ such that $|\varphi_n(A_n)|>n$, a contradiction with b). This completes the proof.

(1.5). *Let $\mathscr{R}$ be a quasi σ-ring of subsets of S and let Φ be a set of k-triangular and exhaustive functions from $\mathscr{R}$ to $(X, |\ |)$. If for every $A\in\mathscr{R}$ the set $\Phi(A)$ is bounded, then $\Phi(\mathscr{R})$ is bounded.*

Suppose that $\Phi(\mathscr{R})$ is not bounded. Then, by (1.4), there exist a sequence $(\varphi_n)_{n\in\mathbf{N}}$ of functions of Φ and a disjoint sequence $(A_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{R}$ such that for every infinite subset M of $\mathbf{N}$ the set $\bigcup_{n\in M}\{\varphi_n(A_n)\}$ is not bounded.

Let now, by (1.3), $(A_{n_i})_{i\in\mathbf{N}}$ be a subsequence of $(A_n)_{n\in\mathbf{N}}$ and $\mathscr{S}$ a quasi σ-ring contained in $\mathscr{R}$ such that $A_{n_i}\in\mathscr{S}$ $(\forall i\in\mathbf{N})$ and the restriction of φ_n to $\mathscr{S}$ is order continuous, for each $n\in\mathbf{N}$.

Let p_1 be a positive real number and let $i_1\in\mathbf{N}$ such that

$$|\varphi_{n_{i_1}}(A_{n_{i_1}})|>2p_1;$$

$\varphi_{n_{i_1}}$ being exhaustive, we can find $h_1>i_1$ such that

$$|\varphi_{n_{i_1}}(A_{n_m})|<p_1/2k^3 \quad \forall m>h_1.$$

We write, $\forall i\in\mathbf{N}$, $\alpha_i=k^2\sup_{\varphi\in\Phi}|\varphi(A_{n_i})|<+\infty$ and we put $p_2=\max\{2p_1, \alpha_{i_1}\}$. Then there exist $i_2>h_1$ and $h_2>i_2$ such that

$$|\varphi_{n_{i_2}}(A_{n_{i_2}})|>3p_2$$

and

$$|\varphi_{n_{i_1}}(A_{n_m})|\leqq p_1/2^2k^3, \quad |\varphi_{n_{i_2}}(A_{n_m})|\leqq p_1/2^2k^3 \quad \forall m\geqq h_2.$$

Similarly, if we write $p_s=\max\{sp_{s-1}, \alpha_{i_{s-1}}\}$ for each $s\in\mathbf{N}$, $(s>1)$ we can find $h_{s-1}<i_s<h_s$ such that

$$|\varphi_{n_{i_s}}(A_{n_{i_s}})|\geqq (s+1)p_s$$

and, for each $r\in\{1, \dots, s\}$,

$$|\varphi_{n_{i_r}}(A_{n_m})|<p_1/2^sk^3 \quad \forall m\geqq h_s.$$

Let now $(A_{n_{i_{s_q}}})_{q\in\mathbf{N}}$ be a subsequence of $(A_{n_{i_s}})_{s\in\mathbf{N}}$ such that $A_0=\bigcup_{q\in\mathbf{N}} A_{n_{i_{s_q}}}\in\mathscr{S}$; we obtain $(\forall q>1)$ from (1.2)

$$|\varphi_{n_{i_{s_q}}}(A_0)|\geqq|\varphi_{n_{i_{s_q}}}(A_{n_{i_{s_q}}})|-k^2\sum_{l<q}|\varphi_{n_{i_{s_q}}}(A_{n_{i_{s_l}}})|-k^3\sum_{l>q}|\varphi_{n_{i_{s_q}}}(A_{n_{i_{s_l}}})|\geqq$$
$$\geqq(s_q+1)p_{s_q}-\sum_{l<q}\alpha_{i_{s_l}}-\sum_{l>q}p_1/2^{s_l-1}\geqq(s_q+1)p_{s_q}-\sum_{l<q}p_{s_l+1}-p_1\geqq$$
$$\geqq s_q p_{s_q}-(q-1)p_{s_{q-1}}\geqq qp_1,$$

a contradiction with the boundedness of $\Phi(A_0)$.

(1.6). *Let $\mathscr{R}$ be a ring of subsets of S with the Subsequential Interpolation Property*[9]) *and let Φ be a set of k-triangular and exhaustive functions from $\mathscr{R}$ to $(X,|\ |)$. If for every $A\in\mathscr{R}$ the set $\Phi(A)$ is bounded, then $\Phi(\mathscr{R})$ is bounded.*

We have to prove that b) of (1.4) is verified. Let $(\varphi_n)_{n\in\mathbf{N}}$ be a sequence of functions of Φ and $(A_n)_{n\in\mathbf{N}}$ a disjoint sequence of sets of Φ.

It is easy to prove that $\mathscr{N}=\{A\in\mathscr{R}:\ \tilde\varphi_n(A)=0\ \ \forall n\in\mathbf{N}\}$ is an ideal of $\mathscr{R}$[10]) and $\mathscr{R}/\mathscr{N}$ satisfies the countable chain condition[11]); therefore by the (7.1.1) of [28] $\mathscr{R}/\mathscr{N}$ is a quasi σ-ring.

Now, we denote for each $n\in\mathbf{N}$ by $\hat\varphi_n$ the function

$$[A]\in\mathscr{R}/\mathscr{N}\to|\varphi_n(A)|$$

and we note that, $\forall n\in\mathbf{N}$, $\hat\varphi_n$ is a k-triangular and exhaustive function from $\mathscr{R}/\mathscr{N}$ to $\mathbf{R}^+$[12]).

Therefore, by (1.5) the set $\bigcup_{n\in\mathbf{N}}\{\varphi_n(A_n)\}\subseteqq\bigcup_{n\in\mathbf{N}}\hat\varphi_n(\mathscr{R}/\mathscr{N})$ is bounded. The proof is complete.

Remark 1. We remark that (1.4) contains Theorem 1, p. 30 of [16] and (1.6) contains the Nikodym's boundedness Theorem of [20], Theorem N of [18], Corollaries 4, 5, 6 p. 29 of [16].

We remark also that from (1.6) we obtain Corollary (Nikodym) of [1] and Theorem 2 of [21][13]).

2. We shall denote below by $\mathscr{A}$ a field of subsets of S and bv $\mathscr{F}$ and $\mathscr{G}$ two lattices contained in $\mathscr{A}$ such that $S\setminus F\in\mathscr{G}$, for each $F\in\mathscr{F}$.

Let φ be a function from $\mathscr{A}$ to $(X,|\ |)$; we say that φ is *inner regular (with respect to $\mathscr{F}$)* in A, $A\in\mathscr{A}$, if for every $\varepsilon>0$ there exists $F\in\mathscr{F}$ such that $F\subseteqq A$ and $\tilde\varphi(A\setminus F)<\varepsilon$.

We say that φ is *inner regular (with respect to $\mathscr{F}$)* on $\mathscr{H}$, $\mathscr{H}\subseteqq\mathscr{A}$, if φ is inner regular (with respect to $\mathscr{F}$) in each $A\in\mathscr{H}$; φ is said *inner regular* if it is inner regular on $\mathscr{A}$.

We note that:

(2.1). *Let φ be a function from $\mathscr{A}$ to $(X, |\ |)$ such that the function*

$$A \in \mathscr{R} \to |\varphi(A)|$$

is k-subadditive. Then φ is inner regular if and only if it satisfies the condition

(°) *For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$ there exist $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F \subseteq A \subseteq G$ and $\tilde{\varphi}(G \setminus F) < \varepsilon$*[14]).

It follows easily from the properties of $\tilde{\varphi}$[15]).

Remark 2. This proposition is valid, in particular, for an inner regular k-subadditive function defined on $\mathscr{A}$ with values in $\mathbf{R}^+$.

If S is a Hausdorff locally compact topological space, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets of S, $\mathscr{A}$ is a field containing $\mathscr{G}$, a function φ from $\mathscr{A}$ to $(X, |\ |)$, such that the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

is k-subadditive, is inner regular (with respect to $\mathscr{F}$) iff the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

is regular (R), in the sense of [8].

(2.2). *Let $\mathscr{F}$ be a semicompact lattice, so a lattice with the property:*

($*$) *For every sequence $(F_n)_{n \in \mathbf{N}}$ in $\mathscr{F}$ such that $\bigcap_{n \in \mathbf{N}} F_n = \emptyset$, there exists $n_0 \in \mathbf{N}$ such that $\bigcap_{n \leqq n_0} F_n = \emptyset$.*

Let φ be an inner regular (with respect to $\mathscr{F}$) function from $\mathscr{A}$ to $(X, |\ |)$; then,

1) *if the function*

$$A \in \mathscr{A} \to |\varphi(A)|$$

is k-subadditive, φ is order continuous and therefore the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

and the semivariation of φ, $\tilde{\varphi}$, are also countably k-subadditive;

2) *if φ is a k-triangular function, φ is $\mathscr{H}$-exhaustive, for every lattice $\mathscr{H} \subseteq \mathscr{A}$ such that for every disjoint sequence $(H_n)_{n \in \mathbf{N}}$ in $\mathscr{H}$ the σ-ring generated by $\{H_n, n \in \mathbf{N}\}$ is contained in $\mathscr{A}$.*

To prove 1), by (1.2), it suffices to prove that φ is order continuous. For this, if $(A_n)_{n \in \mathbf{N}}$ is a decreasing sequence of sets of ' such that $\bigcap_{n \in \mathbf{N}} A_n = \emptyset$, for any $\varepsilon > 0$ and $n \in \mathbf{N}$, let $F_n \in \mathscr{F}$ such that

$$F_n \subseteq A_n \quad \text{and} \quad \tilde{\varphi}(A_n \setminus F_n) < \varepsilon / 2^n k.$$

Then by $(*)$ there exists $m_0 \in \mathbf{N}$ such that

$$\bigcap_{i \leqq m} F_i = \emptyset \quad \forall m \geqq m_0;$$

hence for each $m \geqq m_0$,

$$|\varphi(A_m)| \leqq \tilde{\varphi}(A_m) = \tilde{\varphi}\big(A_m \setminus \bigcap_{i \leqq m} F_i\big) = \tilde{\varphi}\big(\bigcup_{i \leqq m} (A_i \setminus F_i)\big) \leqq$$

$$\leqq \tilde{\varphi}(A_1 \setminus F_1) + k \sum_{1 < i \leqq m} \tilde{\varphi}(A_i \setminus F_i) \leqq k \sum_{n \in \mathbf{N}} \varepsilon/2^n k = \varepsilon.$$

To prove 2), it suffices to remark that, by 1), φ is order continuous and, if $(H_n)_{n \in \mathbf{N}}$ is a disjoint sequence in $\mathscr{H}$, for every $n \in \mathbf{N}$,

$$|\varphi(H_n)| \leqq \big|\varphi\big(\bigcup_{i \geqq n} H_i\big)\big| + k \big|\varphi\big(\bigcup_{i \geqq n+1} H_i\big)\big|.$$

This completes the proof.

(2.3). *Let $\mathscr{F}$ and $\mathscr{G}$ satisfy the property:*

$(**)$ *For each $F \in \mathscr{F}$ and for each sequence $(G_n)_{n \in \mathbf{N}}$ in $\mathscr{G}$ such that $F \subseteqq \bigcup_{n \in \mathbf{N}} G_n$ there exists $n_0 \in \mathbf{N}$ such that $F \subseteqq \bigcup_{n \leqq n_0} G_n$*[16]*), and let $\mathscr{G}$ be closed under the countable union of mutually disjoint sets.*

If φ is a function from $\mathscr{A}$ to $(X, |\ |)$ inner regular (with respect to $\mathscr{F}$) on $\mathscr{G}$, the semivariation of φ, $\tilde{\varphi}$, (and therefore φ) is $\mathscr{G}$-exhaustive.

Let $(G_n)_{n \in \mathbf{N}}$ be a disjoint sequence in $\mathscr{G}$. For every $\varepsilon > 0$, let $F \in \mathscr{F}$ such that

$$F \subseteqq \bigcup_{n \in \mathbf{N}} G_n \quad \text{and} \quad \tilde{\varphi}\big(\bigcup_{n \in \mathbf{N}} G_n \setminus F\big) < \varepsilon;$$

hence, if $n_0 \in \mathbf{N}$ is such that

$$F \subseteqq \bigcup_{n \leqq n_0} G_n$$

for every $m \geqq n_0 + 1$

$$|\varphi(G_m)| \leqq \tilde{\varphi}(G_m) \leqq \tilde{\varphi}\big(\bigcup_{n \in \mathbf{N}} G \setminus F\big) < \varepsilon;$$

the proof is complete.

We remark also:

(2.4). *Let φ be a k-triangular function inner regular (with respect to $\mathscr{F}$) defined on $\mathscr{A}$ with values in $(X, |\ |)$. Then φ satisfies the condition*

$(^{\circ\circ})$ *For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$, there exist $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F \subseteqq A \subseteqq G$ and for every $A' \in \mathscr{A}$ such that $F \subseteqq A' \subseteqq G$ we have*

$$\big||\varphi(A)| - |\varphi(A')|\big| < \varepsilon.$$

Let $A \in \mathscr{A}$ and let $\varepsilon > 0$. Then, by (2.1), we can find $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that

$$F \subseteqq A \subseteqq G \quad \text{and} \quad |\varphi(B)| < \varepsilon/2k \quad \forall B \subseteqq G \setminus F.$$

If $A' \in \mathscr{A}$ and $F \subseteqq A' \subseteqq G$, then obviously

$$(A \setminus A') \cup (A' \setminus A) \subseteqq G \setminus F$$

and therefore

$$\big||\varphi(A)| - |\varphi(A')|\big| \leqq k\,|\varphi(A \setminus A')| + k\,|\varphi(A' \setminus A)| < \varepsilon.$$

In particular, we have:

Corollary (2.4). *Let φ be a k-triangular function defined on $\mathscr{A}$ with values in $\mathbf{R}^+$. If φ is inner regular, φ satisfies the condition*

($\genfrac{}{}{0pt}{}{\circ}{\circ}\circ$) *For every $A \in \mathscr{A}$ and for every $\varepsilon > 0$, there exist $F \in \mathscr{F}_A$ and $A \subseteqq G \in \mathscr{G}$ such that for every $A \in \mathscr{A}$ such that $F \subseteqq A' \subseteqq G$ we have*

$$|\varphi(A) - \varphi(A')| < \varepsilon.$$

Remark 3. If S is a Hausdorff locally compact topological space, $\mathscr{A}$ is the σ-field of the Borel sets of S, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets of S, from (2.4) (resp. from Corollary (2.4)) we obtain that, if φ is an inner regular k-triangular function from $\mathscr{A}$ to $(X, |\ |)$ (resp. to $\mathbf{R}^+$), the function

$$A \in \mathscr{A} \to |\varphi(A)| \in \mathbf{R}^+$$

(resp. φ) is regular on $\mathscr{A}$ in the sense of [7], p. 303; see also Remark 2.

(2.5). *Let $(\Gamma, |\ |)$ be a quasi-normed abelian group and let φ a finitely additive function from $\mathscr{A}$ to $(\Gamma, |\ |)$. Then φ is inner regular if and only if, for every $A \in \mathscr{A}$ and for every $\varepsilon > 0$ there exist $F \in \mathscr{F}$ and $G \in \mathscr{G}$ such that $F \subseteqq A \subseteqq G$ and for every $A' \in \mathscr{A}$ with $F \subseteqq A' \subseteqq G$ we have*

$$|\varphi(A) - \varphi(A')| < \varepsilon.$$

Obviously we can use the same arguments of the proof of Prop. 1, p. 304 of [7].

We say that a function φ from $\mathscr{A}$ to $(X, |\ |)$ is *regular* if

(a) φ is inner regular,

(b) for every $F \in \mathscr{F}$ and for every $\varepsilon > 0$ there exist $E \in \mathscr{G}$, $H \in \mathscr{F}$, $G \in \mathscr{G}$ such that $F \subseteqq E \subseteqq H \subseteqq G$ and $\tilde{\varphi}(G \setminus F) < \varepsilon$[17]).

Remark 4. If we suppose that $\mathscr{F}$ and $\mathscr{G}$ have the property:

($\bullet$)for every $F \in \mathscr{F}$ and for every $G \in \mathscr{G}$ such that $F \subseteqq G$, there exist $E \in \mathscr{G}$, $H \in \mathscr{F}$, such that $F \subseteqq E \subseteqq H \subseteqq G$, clearly a function φ from $\mathscr{A}$ to $(X, |\ |)$ such that the function

$$A \in \mathscr{A} \to |\varphi(A)|$$

is k-subadditive (in particular a k-triangular function) is regular iff it is inner regular (see (2.1)).

In particular, if S is a Hausdorff locally compact (resp. normal) topological space, $\mathscr{F}$ is the lattice of the compact (resp. closed) sets of S, $\mathscr{G}$ is the lattice of the open sets of S, $\mathscr{A}$ is a field containing $\mathscr{G}$, a k-triangular function is regular iff it is inner regular.

We remark also that in the case S Hausdorff locally compact topological space, $\mathscr{A}$ the σ-field of the Borel sets of S, $\mathscr{F}$ the lattice of the compact sets, $\mathscr{G}$ the lattice of the open sets of S, a k-triangular function φ from $\mathscr{A}$ to $(X, |\ |)$ with regular variation[18]) (regular in the sense of [7], p. 303) satisfies the condition ($\bullet$) of the (2.1) and therefore it is regular (see also [24], Theorem 1 and Corollary 1).

3. (3.1) *Let Φ be a set of k-triangular inner regular functions from $\mathscr{A}$ to $(X, |\ |)$. Then, for every $A \in \mathscr{A}$ such that $\Phi(\mathscr{A}_A)$ is not bounded and for every $n \in \mathbf{N}$ there exists $(\varphi, B) \in \Phi \times ((\mathscr{F} \cup \mathscr{G}) \cap A)$ such that*

$$|\varphi(B)| > n \quad \textit{and} \quad \Phi(\mathscr{A}_B) \quad \textit{is} \quad \textit{not bounded.}$$

Assume that there exist $A_0 \in \mathscr{A}$ such that $\Phi(\mathscr{A}_{A_0})$ is not bounded and $n_0 \in \mathbf{N}$ such that for every $(\varphi, B) \in \Phi \times ((\mathscr{F} \cup \mathscr{G}) \cap A_0)$

$$|\varphi(B)| > n_0 \quad \text{implies that} \quad \Phi(\mathscr{A}_B) \quad \text{is bounded.}$$

Let now $F \in \mathscr{F}_{A_0}$ and $\bar{\varphi} \in \Phi$ such that $|\bar{\varphi}(F)| > (1+k)n_0$; therefore we have

$$F \in \mathscr{F}_{A_0}, \quad A_0 \setminus F \in \mathscr{G} \cap A_0, \quad \bar{\varphi}(F) > n_0, \quad \bar{\varphi}(A_0 \setminus F) > n_0.$$

Then, both $\Phi(\mathscr{A}_F)$ and $\Phi(\mathscr{A}_{A_0 \setminus F})$ are bounded, a contradiction with the assumption that $\Phi(\mathscr{A}_{A_0})$ is not bounded.

Now we can give the proof of:

(3.2). *Let Φ be a set of k-triangular and regular functions from $\mathscr{A}$ to $(X, |\ |)$ such that*

α) for every $G \in \mathscr{G}$, $\Phi(G)$ is bounded,

β) for every sequence $(\varphi_n)_{n \in \mathbf{N}}$ of functions of Φ and for every disjoint sequence $(G_n)_{n \in \mathbf{N}}$ of sets of $\mathscr{G}$, there exists an infinite subset M of $\mathbf{N}$ such that $\bigcup_{n \in \mathbf{M}} \{\varphi_n(G_n)\}$ is bounded.

Then $\Phi(\mathscr{A})$ is bounded.

Assume that $\Phi(\mathscr{A})$ is not bounded.

We will show firstly that Φ satisfies the following property:

γ) For every $A \in \mathscr{G}$ such that $\Phi(\mathscr{A}_A)$ is not bounded and for every $n \in \mathbf{N}$ there exists $(\bar{\varphi}, G, A') \in \Phi \times \mathscr{G}_A \times \mathscr{G}_A$ such that

$$(***) \qquad |\bar{\varphi}(G)| > n, \quad G \cap A' = \emptyset, \quad \Phi(\mathscr{A}_{A'}) \quad \text{is not bounded.}$$

Let $A \in \mathscr{G}$ such that $\Phi(\mathscr{A}_A)$ is not bounded and let $n \in \mathbf{N}$; let $h \in \mathbf{N}$ such that $|\varphi(A)| \leqq h$, for every $\varphi \in \Phi$.

There are two possibilities.

Case I: There exists $(\varphi_*, H) \in \Phi \times (\mathscr{F} \cap A)$ such that

$$|\varphi_*(H)| > 2(h+kn) \quad \text{and} \quad \Phi(\mathscr{A}_H) \quad \text{is not bounded.}$$

In this case, let $G' \in \mathscr{G}$, $F' \in \mathscr{F}$, $G'' \in \mathscr{G}$ such that

$$H \subseteqq G' \cap A \subseteqq F' \cap A \subseteqq G'' \cap A \quad \text{and} \quad \tilde{\varphi}_*(G'' \cap A \setminus H) < (k+hn)/k,$$

so $|\varphi_*(F' \cap A)| > h+kn$. Then, if we put

$$G = A \setminus F' \cap A, \quad A' = G' \cap A$$

it is easy to see that $(\varphi_*, G, A') \in \Phi \times \mathscr{G}_A \times \mathscr{G}_A$ verifies $(***)$.

Case II: For every $(\varphi, H) \in \Phi \times (\mathscr{F} \cap A)$, $|\varphi(H)| > 2(h+kn)$ implies that $\Phi(\mathscr{A}_H)$ is bounded.

In this case, let $F \in \mathscr{F}_A$ and $\varphi_* \in \Phi$ such that $|\varphi_*(F)| > 4(h+kn)$; let $G' \in \mathscr{G}$, $F' \in \mathscr{F}$, $G'' \in \mathscr{G}$ such that

$$F \subseteqq G' \subseteqq F' \subseteqq G'' \quad \text{and} \quad \tilde{\varphi}_*(G'' \setminus F) < (h+kn)/k,$$

so

$$|\varphi_*(F' \cap A)| > 2(h+kn) \quad \text{and} \quad |\varphi_*(G' \cap A)| > 2(h+kn).$$

Finally, if we put

$$G = G' \cap A, \quad A' = A \setminus F' \cap A,$$

obviously (φ_*, G, A') verifies the $(***)$.

It is clear now that, by the same argument as that of (3.2) of [14], we obtain a sequence $(\varphi_n)_{n \in \mathbf{N}}$ of functions of Φ and a disjoint sequence $(G_n)_{n \in \mathbf{N}}$ in $\mathscr{G}$ such that $|\varphi_n(G_n)| > n$, for every $n \in \mathbf{N}$; a contradiction with β).

(3.3) *Let $\mathscr{G}$ be a SIP-lattice*[19]) *and let Φ be a set of k-triangular functions from $\mathscr{A}$ to $(X, |\ |)$, $\mathscr{G}$-exhaustive and regular, such that for every $G \in \mathscr{G}$, $\Phi(G)$ is bounded; then $\Phi(\mathscr{A})$ is bounded.*

It suffices to prove that Φ satisfies contion β) of (2.2).

For this, let $(\varphi_n)_{n \in \mathbf{N}}$ be a sequence of functions of Φ and let $(G_n)_{n \in \mathbf{N}}$ be a disjoint sequence of sets of $\mathscr{G}$. We denote respectively by $(G_{n_i})_{i \in \mathbf{N}}$ and by $\mathscr{S}$ a subsequence of $(G_n)_{n \in \mathbf{N}}$ and a ring with the Subsequential Interpolation Property contained in $\mathscr{G}$, such that $G_{n_i} \in \mathscr{S}$ for every $i \in \mathbf{N}$.

Clearly the restriction of φ_n to $\mathscr{S}$ is exhaustive for each $n \in \mathbf{N}$, and the set $\bigcup_{n \in \mathbf{N}} \{\varphi_n(G)\}$ is bounded for each $G \in \mathscr{G}$; therefore, by the (1.6) the set $\bigcup_{i \in \mathbf{N}} \varphi_{n_i}(G_{n_i}) \subseteqq \subseteqq \bigcup_{n \in \mathbf{N}} \varphi_n(\mathscr{S})$ is bounded.

The proof is complete.

Corollary (3.4). *Let $\mathscr{G}$ be a SIP-lattice and suppose that $\mathscr{F}$ and $\mathscr{G}$ have the property:*

($\bullet$) *for every $F \in \mathscr{F}$ and every $G \in \mathscr{G}$ such that $F \subseteqq G$, there exist $E \in \mathscr{G}$, $H \in \mathscr{F}$ such that $F \subseteqq E \subseteqq H \subseteqq G$.*

If Φ is a set of k-triangular, $\mathscr{G}$-exhaustive and inner regular functions from $\mathscr{A}$ to $(X, | \, |)$ such that for every $G \in \mathscr{G}$ $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.

It follows immediately from (3.3) (see Remark 4). In particular we have:

Corollary (3.5). *Let S be a normal topological space, $\mathscr{G}$ the lattice of the open sets, $\mathscr{F}$ the lattice of the closed sets of S, $\mathscr{A}$ a field containing $\mathscr{G}$. If Φ is a set of k-triangular, $\mathscr{G}$-exhaustive and inner regular functions from $\mathscr{A}$ to $(X, | \, |)$ such that for every $G \in \mathscr{G}$ $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

Corollary (3.6). *Let S be a Hausdorff locally compact topological space, $\mathscr{F}$ the lattice of the compact sets, $\mathscr{G}$ the lattice of the open sets, $\mathscr{A}$ a field containing $\mathscr{G}$. If Φ is a set of k-triangular and inner regular functions from $\mathscr{A}$ to $(X, | \, |)$, such that, for every $G \in \mathscr{G}$, $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

It follows immediately from Corollary (3.4) and (2.2).

Corollary (3.7). *Let S be a Hausdorff topological space, $\mathscr{G}$ the lattice of the open sets, $\mathscr{F}$ the lattice of the compact sets of S, $\mathscr{A}$ a field containing $\mathscr{G}$. If Φ is a set of k-triangular and regular functions from $\mathscr{A}$ to $(X, | \, |)$ such that for every $G \in \mathscr{G}$ $\Phi(G)$ is bounded, then $\Phi(\mathscr{A})$ is bounded.*

It follows immediately from (3.3) and (2.2).

Remark 5. Clearly (see (2.1) and Remark 4), Corollary (3.6) contains Theorem 2 and Theorem 3 of [24] (see also [23], [6] Proposition 9, [2] Remark 2, p. 168).

We note that, if we put $\mathscr{F} = \mathscr{G} = \mathscr{A}$, (3.3) yields a Nikodym's boundedness theorem for k-triangular functions defined in a field which is a SIP-lattice. Moreover, from (3.3) we can obtain a Dieudonné boundedness type theorem for finitely additive functions from $\mathscr{A}$ with values in a topological commutative group Γ (see [14]). In fact, if Γ is a topological commutative group with neutral element 0, a finitely additive function φ from $\mathscr{A}$ to Γ is $\mathscr{G}$-exhaustive (resp. inner regular, regular (in the sense of [14])) iff, for every continuous real-valued quasi-norm ϱ on Γ, the $\mathbf{R}^+$-valued 1-triangular function $\varrho \circ \varphi$ is $\mathscr{G}$-exhaustive (resp. inner regular, regular)[20]).

Therefore:

Corollary (3.8). *Let Γ be a topological commutative group and let $\mathscr{G}$ be a*

SIP-lattice. If Φ is a set of finitely additive and $\mathcal{G}$-exhaustive regular functions from $\mathcal{A}$ to Γ, such that for every $G\in\mathcal{G}$, $\Phi(G)$ is $\mathcal{U}$-bounded, then $\Phi(\mathcal{A})$ is $\mathcal{U}$-bounded[21]).

For every continuous real-valued quasi-norm ϱ on Γ, apply (3.3) to the set $\bigcup_{\varphi\in\Phi}(\varrho\circ\varphi)$.

Notes

[1]) A function ψ from $\mathcal{R}$ to $\overline{\mathbf{R}}^+$ is said *k-subadditive* (resp. *countably k-subadditive*) if, for any disjoint sets A, B from $\mathcal{R}$, $\psi(A\cup B)\leqq\psi(A)+k\psi(B)$ (resp. for any disjoint sequence $(A_n)_{n\in\mathbf{N}}$ in $\mathcal{R}$ such that $\bigcup_{n\in\mathbf{N}}A_n\in\mathcal{R}$, $\psi\left(\bigcup_{n\in\mathbf{N}}A_n\right)\leqq\psi(A_1)+k\sum_{n>1}\psi(A_n)$) (see [15], [16]).

[2]) See [15], Corollary 1 for the case φ k-triangular with values in an abelian quasi-normed group and [19], Corollary (2.3) for the case φ finitely additive.

[3]) In fact, for each $A\in\mathcal{R}$,

$$|\varphi(A)|\leqq|\varphi(A\cap A_0)|+k|\varphi(A\setminus A_0)|;$$

therefore $\varphi(\mathcal{R}_{A_0})$ bounded implies $\varphi(\mathcal{R})$ bounded.

[4]) See [15], Lemma 2, for the case φ k-triangular function with values in an abelian quasi-normed group.

[5]) For the definition of quasi σ-ring see [3], [9], [13], [28]; see also [25], Lemma 1.

[6]) We note that function a φ from $\mathcal{R}$ to $(X, |\ |)$ is exhaustive iff for every disjoint sequence $(A_n)_{n\in\mathbf{N}}$ of set of $\mathcal{R}$

$$\lim_n\varphi(A_n\cap A)=0,$$

uniformly with respect to $A\in\mathcal{R}$ (see the proof of (1.1), Ch. II of [4]) see also note [10]), p. 134 of [4].

[7]) It follows from (1.1) of [13]; in fact, it is easy to see that it is true also for an exhaustive function η from $\mathcal{R}$ to $\mathbf{R}^+$ such that

$$\eta(X)\leqq\eta(Y)\quad\text{if}\quad X,Y\in\mathcal{R}\colon\ X\subseteqq Y.$$

[8]) We refer to [16], remark p. 29, for an example of a set Φ of (real) 1-triangular exhaustive functions verifying a), for which the set $\Phi(\mathcal{R})$ is not bounded. We write $\Phi(A)=\bigcup_{\varphi\in\Phi}\varphi(A)$ $\forall A\in\mathcal{R}$ and $\Phi(\mathcal{H})=\bigcup_{A\in\mathcal{H}}\Phi(A)$ $\forall\mathcal{H}\subseteqq\mathcal{R}$.

[9]) See [10], [5] for the definition of rings with the Subsequential Intersolation Property (rings with the (P2) property in [28], satisfying condition (E_2) in [9]).

[10]) We note that, for each $n\in\mathbf{N}$, $\{A\in\mathcal{R}\colon\ \tilde{\varphi}_n(A)=0\}$ is an ideal of $\mathcal{R}$; see also [17], [27].

[11]) In fact, let $\mathcal{A}$ be a disjoint set of non-zero elements of $\mathcal{R}/\mathcal{N}$; we write, $\forall\,(n,k)\in\mathbf{N}\times\mathbf{N}$, $\mathcal{A}_k^{(n)}=\{[A]\in\mathcal{R}/\mathcal{N}\colon\ \tilde{\varphi}_n(A)>1/k\}$. Then $\mathcal{A}=\bigcup_{(n,k)\in\mathbf{N}\times\mathbf{N}}\mathcal{A}_k^{(n)}$ and, φ_n being exhaustive $\forall n\in\mathbf{N}$, $\mathcal{A}_k^{(n)}$ is or empty or finite set, $\forall(n,k)\leqq\mathbf{N}\times\mathbf{N}$.

[12]) In fact, $\forall n\in\mathbf{N}$, we have $|\varphi_n(A)|=|\varphi_n(B)|$ if $[A]=[B]$, $\hat{\varphi}_n([\emptyset])=0$,

$|\hat{\varphi}_n([A])-\hat{\varphi}_n([B])| \leqq k\varphi_n(A\setminus B)+k\varphi_n(B\setminus A) = k\hat{\varphi}_n([A]\setminus[B])+k\hat{\varphi}_n([B]-[A])$ $\forall[A], [B]\in\mathscr{R}/\mathscr{N}$; for every disjoint sequence $([A_p])_{p\in\mathbf{N}}$ we put $A_1'=A_1$ and, $\forall p>1, A_p'=A_p-\bigcup_{i<p} A_p\cap A_i$ and we have

$$\lim_p \hat{\varphi}_n([A_p]) = \lim_p |\varphi_n(A_p')| = 0.$$

[13]) We note that, if $\mathscr{R}$ is a σ-ring, a k-triangular and order continuous function is exhaustive. Moreover, if X is a commutative semigroup with a family F of non-negative real valued functions f which have the property

$$f(x)-f(y) \leqq f(x+y) \leqq f(x)+f(y), \quad \text{for each} \quad x, y\in X,$$

for every triangle set function ([21]) order continuous μ from $\mathscr{R}$ to $(X, |\ |)$ the function

$$\nu\colon\ A\in\mathscr{R} \to f(\mu(A))\in[0, +\infty[$$

is, for every $f\in F$, a 1-triangular and order continuous function.

[14]) If S is a Hausdorff locally compact topological space, $\mathscr{A}$ is the σ-field of the Borel sets of S, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets of S, the (°) is the condition (R) of [23], [24].

[15]) For every $A\in\mathscr{A}$ and for every $\varepsilon>0$, let $F\in\mathscr{F}_A$ and $H\in\mathscr{F}_{S\setminus A}$ such that $\tilde{\varphi}(A\setminus F)<\varepsilon/2$ and $\tilde{\varphi}(S\setminus A\setminus H)<\varepsilon/2$ and put $G=S\setminus H$.

[16]) For instance, if S is a Hausdorff topological space, $\mathscr{A}$ is the σ-field of the Borel sets of S, $\mathscr{F}$ and $\mathscr{G}$ are respectively the lattice of the compact sets and the lattice of the open sets, $\mathscr{F}$ and $\mathscr{G}$ satisfy the property $(**)$ (and therefore $\mathscr{F}$ has the $(*)$).

[17]) If $(X, |\ |)$ is a quasi-normed group and φ is a finitely additive function from $\mathscr{A}$ to $(X, |\ |)$, this is the definition of regular finitely additive function of [14].

[18]) The variation $|\varphi|$ of φ is defined in the usual way;

$$|\varphi|(A) = \sup_{\Pi} \sum_{B\in\Pi} |\varphi(B)| \quad A\in\mathscr{A},$$

where the supremum is taken over all partitions of A into a finite number of disjoint sets in $\mathscr{A}$.

[19]) We say that a lattice $\mathscr{G}$ is a SIP-lattice if for each disjoint sequence $(G_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{G}$ there exist a subsequence $(G_{n_i})_{i\in\mathbf{N}}$ of $(G_n)_{n\in\mathbf{N}}$ and a ring $\mathscr{S}$ with the SIP contained in $\mathscr{G}$, such that $G_{n_i}\in\mathscr{S}$, for each $i\in\mathbf{N}$ ([14]).

[20]) See [14] for the definitions of finitely additive inner regular and regular functions from $\mathscr{A}$ to Γ. Recall that for every neighbourhood U of 0, there exist an $\varepsilon>0$ and a continuous (real-valued) quasi-norm ϱ on Γ such that $\{x\in\Gamma\colon\ \varrho(x)<\varepsilon\}\subseteqq U$.

[21]) See [14] for the definition of $\mathscr{U}$-bounded subset of Γ; recall that a subset Y of Γ is $\mathscr{U}$-bounded iff, for every continuous real-valued quasi-norm on Γ, $\sup_{y\in Y}\varrho(y)<+\infty$ ([28], Th. (6.8) (a)). See [14], (3.2).

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Bibliographie

Karl-Heinz Becker—Michael Dörffer, Dynamical Systems and Fractals (Computer graphics experiments in Pascal), XII+398 pages, Cambridge University Press, New York—Portchester—Melbourne—Sydney, 1989.

Nowadays nobody is surprised about that the computer enters in almost every area of the life. At the same time one could think of fractals as the things too hard to use PC-s for them. This book proves that the most wide-spread PC-s, such as IBM and Apple for example, can offer some really good results and experiences in the area of chaos and fractals.

The book is very good for two purposes. We can recommend it to all people who have some familiarity with computers and enjoy making nice graphics on screen or printer. The book is so well-illustrated and so various that the reader may not lose his/her interest for years.

A more serious use of the book can be in teaching dynamical systems and fractals. The teacher can demonstrate the result of the rigorous mathematics or can motivate the students. It is advisable to give the book to students for undertaking their own experience in the subject. It is worth noting that the book also has a very good bibliography.

In sum, this book is well-written and has very few misprints, which is very important in the lists of programs.

Á. Kurusa (Szeged)

Jürgen Bokowski—Bernd Sturmfels, Computational Synthetic Geometry, (Lecture Notes in Mathematics, 1359), 168 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

Computational geometry is a rapidly growing young field on the border line of mathematics and computer science. The main object of this volume is the discussion of the algorithmic aspects of certain fundamental realizability problems in discrete geometry.

The authors investigate the algorithmic Steinitz problem: Given a lattice, is it polytopal?; projective incidence theorems: Give an enumeration procedure for all incidence types over a given field; diophantine problems in combinatorical geometry: Given a configuration of points and lines in the plane, can it be constructed with pencil and ruler only?; the embedding of triangulated manifolds. The methods are based on the theory of matroids and oriented matroids. The book is recommended to geometers and computer scientists as an introduction and motivation for further research.

J. Kincses (Szeged)

N. Bourbaki, Elements of Mathematics, Algebra I, XXIII+708 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

N. Bourbaki, Elements of Mathematics, Commutative Algebra, XXIV+625 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

In the late thirties of our century appeared the first volume in the famous series "Éléments de mathematique" by the polycephalic mathematician known as Nicolas Bourbaki. This was a nonexistent Frenchman. The name has simply been appropriated to designate a group of mathematicians, almost exclusively French, who form a sort of cryptic "société anonyme". As an institutional connection N. Bourbaki sometimes used the University of Nancago, a playful reference to the fact, that two of the moving spirit within the group were for a while connected with universities in the Chicago area and Nancy (André Weil and Han Dieudonné).

The first part of the series "Les structures fondamentales de l'analyse" contains half a dozen subheadings: Set Theory, Algebra, General Topology, Functions of Real Variable, Topological Vector Spaces and Integration. It was followed by four subsequent works: Lie Groups and Lie Algebras, Commutative Algebra, Spectral Theory, Differential and Analytic Manifolds.

The presentation of the subject by Bourbaki can be characterized by uncompromising adherence to the axiomatic approach and to a starkly abstract and general form that portrays clearly the logical structure. The text of each book consists of the dogmatic exposition of the theory, therefore it contains in general no references to the literature. Few bibliographical references are gathered together in "Historical Notes" usually at the end of each chapter.

The volume Algebra I contains the first three chapters of the whole one: Algebraic Structures; Linear Algebra; Tensor Algebras, Exterior Algebras, Symmetric Algebras.

The whole Algebra has the following six additional chapters: Polynomials and Rational Fractions, Fields Ordered Groups and Fields, Modules over Principal Ideal Rings, Semi-simple Modules and Rings, Sesquilinear and Quadratic Forms.

The volume Commutative Algebra has seven chapters: Flat Modules, Localization, Graduations, Filtrations and Topologies, Associated Prime Ideals and Primary Decomposition, Integers, Valuations, Divisors.

In both volumes each chapter is followed by exercises.

The works of N. Bourbaki are not easy pieces of reading, but everybody can enjoy them, who likes the strict axiomatic treatment. In my opinion these masterpieces must have places in every good mathematical library.

Lajos Klukovits (Szeged)

Kenneth S. Brown, Buildings, VIII+215 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1989.

This book was written with the explicit aim of offering a profound and systematical introduction to the "theory of buildings". It is a kind of theories which brings hard connection between geometry and group theory. In order to illustrate this connection, it is sufficient to refer some concepts playing an important role constructing buildings: reflection groups, associated simplicial complexes, Coxeter diagram, Coxeter complexes, etc.

Whereas the study of this book does not presume any high level preliminary knowledge, in consequence of the material, readers who are in possession of well-founded knowledges on the classification of algebras and abstract simplicial complexes, have an advantage over the beginners when look for the central ideas of this theory.

These important basic blocks of the theory are also given in the book, each of them is discussed in accordance with desirable depth. In the first chapter the reader get acquainted with geometric imagination (finite reflexion groups) which aid him further on to pursue and approach the theory. After developing the necessary apparatuses (abstract reflection groups and Coxeter complexes in Chapter II and Chapter III) the notion of a building is introduced:

"A building is a simplicial complex Δ which can be expressed as the union of subcomplexes Σ (called apartments) satisfying the following axioms:

(B0) Each apartment Σ is a Coxeter complex.

(B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.

(B2) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise."

The author propounds possible directions for further studies himself, refering Tits' works first of all.

This monograph makes it possible for a beginner getting acquinted with theory of buildings by means of various knowledges arranged in it. The book is also useful for readers well up in the referred branches of geometry and algebra to examine a new interesting theory.

J. Kozma (Szeged)

Komaravolu Chandrasekharan, Classical Fourier Transforms (Universitext), III+172 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

"Again a new book about Fourier transform" — one may think of, but this book rises above the others from many viewpoints.

This self-contained book gives a through introduction to classical Fourier transforms in a very clear and compact form. It only needs a basic knowledge in real and complex analysis.

It is divided into two main parts, the first of which is about the L_1 theory. After the basic properties, the Poisson's summation formula, the central limit theorem, Wiener's general tauberian theorem are given.

In the second part the L_2 theory can be found. Plancherel's theorem, Heisenberg's inequality, the Paley—Wiener theorem, Hardy's beautiful interpolation formula and Bernstein's inequalities are in this part.

At the end of the book a third chapter about the Fourier—Stieltjes transform is placed.

We recommend it not only to undergraduates from almost any area of the technical sciences, physics or mathematics but also to teachers who want to teach the classical Fourier transform in a modern form.

Á. Kurusa (Szeged)

Judith N. Cederberg, A Course in Modern Geometries (Undergraduate Texts in Mathematics), XII+232 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1989.

This new introductionary work is a good possibility for undergraduate students to have a view of the geometries: How are they built? In this way different geometries become palpable even for the beginners.

Chapter 1 presents in a brief and clear manner the structure of some a little complicated finite geometries. It gives an excellent introduction for an inexperienced student to axiomatic way of thinking.

In Chapter 2 a short axiomatic development of the Euclidean plane geometry is given, after that follows a detailed (in the necessary degree) observation of non-Euclidean geometries: hyperbolic geometry and elliptic geometry.

The reader can find a circumstantial discussion of the theory of parallels in the respective geometries.

Chapter 3 covers the treatment of transformation groups in the Euclidean plane. A self-contained section deals with the problems of symmetry groups of the Euclidean plane, furthermore here can be found a brief characterization of the seven frieze groups. As the analytic model of the Euclidean plane is also developed, it is easy to show the matrix representation of transformation groups.

In Chapter 4 — after a short axiomatic introduction and the presentation of infinite model — a clear and well arranged treatment of the real projective plane follows.

A large number of good exercises at the end of each section especially makes the book suitable for using up as a textbook of an undergraduate course.

Each chapter ends with a section proposing further readings regarding the previously discussed subject. In such a way the author can emphasize the introductionary feature of the book on one hand, and the sufficiency of the material for following deeper studies, on the other hand.

J. Kozma (Szeged)

Complex Analysis, Edited by J. Hersch and A. Huber, XII+245 pages, Birkhäuser Verlag, Basel—Boston—Berlin, 1988.

The present volume contains 22 papers dedicated to Albert Pfluger on the occasion of his 80th birthday.

The wide range of the topic of these articles shows Albert Pflugler's strong influence on complex analysts all over the world. His main research is relating to the following fields: entire functions, Riemann surfaces, quasiconformal mappings, schlicht functions. The papers contained in this volume deal with the most important subjects such as conformal and quasiconformal mappings and related extremal problems, Riemann surfaces, meromorphic functions, subharmonic functions, approximation and interpolation.

The collection of these interesting papers gives the reader an insight into the newest results of complex analysis particularly concerning its fields mentioned above.

J. Németh (Szeged)

Edwards B. Davies, Heat Kernels and Spectral Theory (Cambridge tracts in mathematics, 92), IX+197 pages, Cambridge University Press, Cambridge—New York—New Rochelle—Melbourne—Sydney, 1989.

What on Earth can be said to be more classical subject than the heat equation in the theory of second order elliptic operators?! I think nothing is more classical.

But the heat equation at present also is the most recent subject in my opinion. This book is just about the dramatic recent improvements in the quantitative understanding of the heat kernels. This is also shown by its bibliography in which most of the references are dated after 1980.

The author, one of the most known researchers of this subject, considers variable coefficient operators on regions in Euclidean space and the Laplace—Beltrami operators on complete Riemannian manifolds. The most important toole for these investigations is the Grass' theory of logarithmic Sobolev inequalities which yields to ultracontractive bounds. The reader can find some historical notes and supplementary information after each chapter that help the orientation in the subject very well.

We recommend this book to the researchers of this subject as well as to the mathematical physicists.

Á. Kurusa (Szeged)

J. Dieudonné, A History of Algebraic and Differential Topology 1900—1960, XXXI+648 pages, Birkhäuser, Boston—Basel, 1989.

This book describes the main events and results in the expansion of the algebraic and differential topology prior to 1960. There is only one part which is not covered by the text at all, namely, that which is called "low-dimensional topology". In the focus of setting up of the material stay the history on the emergence of ideas and methods which open new fields of research. The text is divided into three parts. The first part presents the various homology and cohomology theorems, the concept of differentiable manifolds. The second part introduces the concept of degree, discusses dimension theory and separation theorems, the fixed point theorems, local homological properties, quotient spaces and their homology, homology groups and homogeneous spaces and at the end of this part applications of homology to geometry and analysis are given. The third part defines homotopy groups and covering spaces, introduces the concept of fibrations and the homology at fibrations, studies the relations between homotopy and homology, investigates the cohomology operations and generalized homology and cohomology.

The familiarity of the reader with elementary algebra and general topology is assumed.

L. Gehér (Szeged)

Differential Games and Applications, Proceedings. Edited by T. S. Basar, P. Bernhard (Lecture Notes in Control and Information Sciences, 119), VII+201 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

The proceedings contain fifteen articles on differential and dynamic games based on the lectures given at the Third International Symposium on Differential Games and Applications held at INRIA, Sophia-Antipolis, France in 1988. Some of the topics involved are: discrete two-person constant-sum dynamic games; zero-sum differential games of the pursuit-evasion type; stochastic games; applications of the nonzero-sum discrete-time dynamic theory, differential game theory in predator-prey system.

The volume covers a large variety of areas and presents recent developments on topics of current interest. It is warmly recommended to researchers in differential and dynamic games, systems and control, operation research and mathematical economics.

L. Hatvani (Szeged)

Differential Geometry and Topology (Lecture Notes in Mathematics, 1369), VI+366 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1989.

The Nankai Institute of Mathematics, Tianjin, PR China held a Special Year in Geometry and Topology during the academic year 1986—87. This volume contains articles written by invited speakers: T. E. Cecil and S. S. Chern: Dupin Submanifolds in Lie sphere Geometry; R. L. Cohen and U. Tillmann: Lectures on Immersion Theory; S. Murakami: Exceptional Simple Lie Groups and Related Topics in Recent Differential Geometry; U. Simon: Dirichlet Problems and the Laplacian in Affine Hypersurface Theory. The other 20 papers give an up-to-date account on the research activity of the participants of this Special Year from PR China in differential geometry. The central theme of these articles is the geometry and topology of submanifolds and immersions.

This volume will be of interest to researchers in this field or on related subjects.

Péter T. Nagy (Szeged)

Dynamical Systems, Proceedings. Edited by J. C. Alexander (Lecture Notes in Mathematics, 1342), VIII+726 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

The Mathematics Department of the University of Maryland devoted its special academic year 1986—1987 to various aspects of Dynamics. They had a great number of visitors during the special year and organized three conferences entitled Ergodic Theory and Topological Dynamics; Symbolic Dynamics and Coding Theory; Smooth Dynamics, Dynamics and Applied Dynamics. These proceedings contain some of the lectures given at the conferences, some achievements of the special year and papers concerned with the questions and problems raised at the conferences. The reader can find articles on such important topics of dynamics as periodicity, ergodicity, strange attractors, chaotic behaviour, Markov shifts, entropy, automata, etc. The space is not enough here to give a complete list of topics but the reviewer can guarantee that the volume helps the reader obtain an insight into the modern theory of dynamical systems and its application.

L. Hatvani (Szeged)

Foundations of Software Technology and Theoretical Computer Science, Edited by Kesav V. Nori (Proceedings of the Seventh Conference in Pune, India, LNCS, 287), IX+540 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1987.

These Proceedings contain 31 selected contributions and five invited talks or, more exactly, two of them plus three rather short abstracts. The lecture "Parallelism and Programming: A Perspective" presented by K. M. Chandy and J. Misra gives a real perspective and a good overview of the topic. The other invited speaker, R. Parikh mixes the thoughts of ancient philosophers with formal models of knowledge and logic in his brilliant mini-essay "Some Recent Applications of Knowledge".

The ordinary papers were organized in the following 9 sessions.

S.1 Automata and Formal Languages; S.2 Graph Algorithms & Goemetric Algorithms; S.3 Distributed Computing; S.4 Parallel Algorithms; S.5 Database Theory; S.6 Logic Programming; S.7 Programming Methodology; S.8 Theory of Algorithms; S.9 Software Technology.

Some results and methods became a bit dated because of the later archivements (e.g. the equivalence problem for n-tape finite automata considered by Culik and Linna has been settled since then). The interested specialist, however, will still find a lot of valuable information and challenging problems in this volume.

J. Virágh (Szeged)

R. V. Gamkrelidze, Analysis I, (Encyclopaedia of Mathematical Sciences, 13), 238 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

The text is divided into three parts. The first part (Series and Integral Representations) develops the theory of numerical and function series and of improper integrations in two directions. One of them is the justification of operations with infinite series and the other is the creation of techniques for using series in the solution of mathematical and applied problems. The second part (Asymptotic Methods in Analysis) starts with giving the simplest examples of asymptotic expansions and the basic ideas behind Laplace's method, the method of stationary phase and some asymptotic estimates for sums and series are also considered. Further asymptotic methods for the solutions of ordinary and partial differential equations and systems are described and asymptotic forms for the solutions of second order equations in the complex domain is constructed. The third part (Integral Transforms) deals with the usual integral transforms especially with the Fourier, Laplace, Mellin, Bessel and Hankel transforms. These are considered not from the point of view of functional analysis, that is not as mappings from a function space into another function space, but from the point of view of the applications to the solutions of ordinary and partial differential equations and of integral equations.

The book is highly recommended to anybody who are interested in the applications of the integral transform methods for solutions of differential equations.

L. Gehér (Szeged)

Geometric Aspects of Functional Analysis, Edited by J. Lindenstrauss and V. D. Milman. (Lecture Notes in Mathematics, 1317, 1376), VI+289 and VI+288 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

These are the third and fourth published volumes of the proceedings of the Israel Seminar on Geometric Aspects of Functional Analysis which cover the 1986—87 and 1987—88 sessions of the seminar. From the volumes it turns out that one of the main objects of the seminar was the systematic study of the connections between finite dimensional convexity theory and Banach space theory. The participants developed a powerful technic based on probabilistic and combinatorial methods to obtain quantitative versions of classical theorems of convexity. It is worth quoting here a typical result of this kind which can be found in the paper of I. Bourgin, J. Lindenstrauss and V. D. Milman "Minkowski sums and symmetrizations". They proved that for every $\varepsilon>0$ we get from K (a given n-dimensional convex body) after performing $c\cdot n\cdot\log n+c(\varepsilon)\cdot n$ random Minkowski symmetrizations a body which is (with a large probability) up to ε an Euclidean ball.

Most of the papers in both volumes are original research papers and contain new and strong methods. The volumes are recommended to research workers in convex geometry and functional analysis.

J. Kincses (Szeged)

Vladislav V. Goldberg, Theory of Multidimensional $(n+1)$-Webs, (Mathematics and Its Applications), XXI+466 pages, Kluwer Academic Publishers, Dordrecht—Boston—London, 1988.

This comprehensive book on web geometry gives a complete treatment of the new development of this theory, which principally was connected with the scientific activity of the author of this book.

The history of differential geometric web theory has three periods. The first one was the work of the school of Wilhelm Blaschke between the two World Wars in Germany. Blaschke and his associates published a series of papers under the heading "Topologische Fragen der Differentialgeometrie" and a monograph: W. Blaschke—G. Bol: "Geometrie der Gewebe". The basic idea of the papers and the book was to apply the theory of local invariants of analytical differential equations to the classification of finite sets of smooth local line families in plane domains, whose abstract analogies appear in the axiomatics of projective planes. These local line structures of the plane (and also line and plane configurations in the 3-space) were called webs of lines or surfaces. The most interesting result of this period was the introduction of local coordinate loops for three families of "parallel" lines (3-webs) in the plane and the translation of geometric closure conditions into algebraic language. These results made an essential influence on the rapid development of abstract geometric algebra, and motivated investigations in non-associative algebra, combinatorial and topological geometry. The second period was initiated by the dissertation of S. S. Chern written under Blaschke in Hamburg (1936), where 3-webs on $2r$-dimensional domains were introduced as $3r$-codimensional foliations in general position. After more than 30 years this research had a new beginning in the series of papers of M. A. Akivis and his school in Moscow. They worked out a complete, closed theory of local 3-webs on $2r$-dimensional manifolds, using Cartan's differential calculus and reduction techniques of principal fiber bundles associated to differential geometric structures. They have clarified the interrelation of this theory to non-associative algebra, algebraic geometry and projective differential geometry.

In the third period the investigations were extended to the theory of more than three foliations. In this case the geometric picture will be, of course, much more complicated, but the theory establishes new connections with other branches of mathematics, especially with algebraic geometry, complex analysis, Abelian differential equations, characteristic classes, etc. The purpose of the present book is to give a systematic explanation of the theory developed in this period, whose essential part consists of the author's own results. The first two Chapters contain a treatment of the differential geometric methods and constructions, used in the study of $(n+1)$-webs, which are defined by $n+1$ foliations of codimension r in general position on a manifold of dimension nr. In Chapter 3 there a treatment of the correspondence between $(n+1)$-webs and local differentiable n-quasigroups is given. Chapter 4 contains the characterization of different classes of $(n+1)$-webs satisfying certain closure conditions. Chapter 5 gives a study of the realization of webs by foliations of Schubert varieties induced by projective surfaces. There is also given a systematic investigation of the case, when the generating projective surfaces are algebraic varieties. These results present a lot of very interesting examples of $(n+1)$-webs, making clear the very close relationship between web geometry and algebraic geometry. Chapter 6 is devoted to the study of application of web theory to the theory of point correspondences of $n+1$ projective spaces and to the theory of holomorphic mappings between polyhedral domains in a complex vector space. Chapter 7 contains a study of webs, which is given by four r-codimensional foliations on a $2r$-dimensional manifold. These webs can be coordinatized by a pair of orthogonal quasigroups. Different closure conditions and their realization as webs on Grassmannians induced by projective or algebraic surfaces are investigated. The last Chapter 8 is devoted to the rank problem of webs formulated by S. S. Chern and P. A. Griffiths in connection with the theory of Abelian differential equations.

This book is a basic reference of web geometry and gives a new impulse to the further development of this theory and of the related fields: non-associative algebra, topological, combinatorial and algebraic geometry, theory of foliations and their applications. It is highly recommended to all mathematicians interested in the interrelations of analytical theory, geometry, algebra and topology.

Peter T. Nagy (Szeged)

Harmonic Analysis and Partial Differential Equations, Edited by J. Garcia-Cuerva (Lecture Notes in Mathematics, 1384), VII+213 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

Since 1979 every four year a conference has been organized on Harmonic Analysis and Partial Differential Equations in El Escorial, Spain. This book contains the courses and lectures held in 1987. The characteristic main courses took three or four hours. Their authors and titles were: D. L. Burkholder: Differential subordination of harmonic functions and martingales; P. W. Jones: Square functions, Cauchy integrals, analytic capacity and harmonic measure; C. E. Kenig: Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation; S. Wainger: Problems in harmonic analysis related to curves and surfaces with infinitely flat points.

Let us cite a few introductory sentences from the first of them to characterize the text: "A fruitful analogy in harmonic analysis is the analogy between a conjugate harmonic function and a martingale transform. One idea that underlies both of these concepts is the idea of differential subordination. Our study of it here yields new information about harmonic functions and martingales, and their interaction. For example, let H be a real or complex Hilbert space with norm $|\cdot|$. Let u and v be harmonic in the open unit disk of C with values in H. If $|v(0)| \leqq |u(0)|$ and $|\nabla v(z)| \leqq |\nabla u(z)|$ for all z with $|z|<1$, then

$$m(\{t\in[0, 2\pi): |u(re^{it})|+|v(re^{it})| \geqq 1\}) \leqq 2 \int_0^{2\pi} |u(re^{it})|\, dt,$$

where $0<r<1$ and m denotes Lebesgue measure."

The second half of the book consists of the ten 45-minute lectures.

The material comprehending more branches of mathematics offers up-to-date results and problems and at the same time by its presentation it is attractive for non specialists as well.

L. Pintér (Szeged)

M. Humi—W. Miller, Second Course in Ordinary Differential Equations for Scientists and Engineers (Universitext), XI+441 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

Nowadays the number of students taking courses in differential equations has been growing in consequence of the important applications of this theory in sciences and engineering. More and more of these students want to get acquainted with the more advanced topics of the theory of differential equations. While there are many good elementary books, there is a need for texts on the advanced topics. This book, which is an outgrowth of second courses in differential equations taught by the authors in the last ten years at Worcester Polytechnic Institute, bridges this gap.

Chapter 0 recalls the method of solutions of differential equations by series. Chapter 1 is devoted to boundary value problems. Chapter 2 treats special functions being solutions of important boundary value problems. Chapter 3 contains some additional material to the theory of linear systems given by beginning texts. Chapter 4 through 10, forming the most attractive and valuable part of the book, examine specific applications of differential equations: Chapter 4: Applications of Symmetry Principles to Differential Equations; Chapter 5: Equations with Periodic Coefficients; Chapter 6: Green's Functions; Chapter 7: Perturbation Theory; Chapter 8: Phase Diagrams and Stability; Chapter 9: Catastrophes and Bifurcations; Chapter 10: Sturmian Theory.

The style of the book is well-guided by the intended audience: it does not contain exhausting proofs of mathematical theorems, they are rather motivated by examples. By the way, the great number of examples make the book easily readable and useful for students and users interested in advanced topics of differential equations and their applications.

L. Hatvani (Szeged)

Robion C. Kirby, The Topology of 4-manifolds (Lecture Notes in Mathematics, 1374), VI+108 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo—Hong Kong, 1989.

Well, the reviewer's task is very hard when the subject of the book is so special, recent and actively studied as the present one. I think the best I can do is to enumerate the main titles of the book not to bore the non-specialists: Rihlin's theorem, Freedman's work, Donaldson's work. $p_1(M)=3\sigma(M)$, Wall's diffeomorphisms and Casson handless.

Anyway the book gives some introduction to the most necessary specialities of the subject as well as some interesting new results. It offers a review on the present state of its subject so it is necessary not only to the researchers who work on this topic but also for those mathematicians and physicists who want to work on or only want to know the main results for possible applications.

Á. Kurusa (Szeged)

P. J. M. van Laarhoven—E. H. L. Aarts, Simulated Annealing: Theory and Applications (Mathematics and its Applications), XI+187 pages, D. Reidel Publishing Company, Dodrecht—Boston—Lancaster—Tokyo, 1987.

The general local search method to solve a combinatorial optimization problem such as the Travelling Salesman Problem is to get iteratively better solutions by searching a neighborhood of the current solution for improvement. The simulated annealing algorithm modifies this approach by introducing two new components: the next solution is selected using randomization and sometimes it is accepted even if it is worse than the previous solution (thus giving, a chance to get out of a local minimum). The name comes from an analogy in physics (annealing is a process of slowly cooling a solid to obtain a low energy state).

Simulated annealing was introduced by Kirkpatrick, Gelatt and Vecchi, and independently by Černy in the early 80's. It was received with great interest as a possible way of obtaining good approximate solutions for notoriously hard computational problems.

The book of van Laarhoven and Aarts gives an overview of the considerable amount of research done already in this field. Theoretically the main problem is to find good "cooling schedules" — the questions arising here concern inhomogeneous Markov chains. The relevant results are formulated without proofs. More emphasis is given to the description of several proposed cooling schedules and their performance in practice. These chapters provide valuable information about the efficiency of the simulated annealing method by comparing it with other known approximation algorithms. A good survey is given of different applications e.g. in computer-aided circuit design. (We note here that a new aspect is described in the book: Simulated Annealing and Boltzmann Machines — A Stochastic Approach to Combinatorial Optimization and Neural Computing, by E. Aarts and J. Korst, Wiley-Interscience Series in Discrete Mathematics and Optimization, 1988.)

The book is a very nice synthesis and it solves the difficult task of presenting a diverse field within 200 pages excellently. It has a clear organization, gives an insightful and balanced picture and it is accessible to non-specialists as well. As simulated annealing is a tool that should be at the disposal of everybody interested in combinatorial optimization, the book is highly recommended as an introduction and guide to this topic.

György Turán (Szeged)

P. Lochak—C. Meunier, Multiphase Averaging for Classical Systems (With Applications to Adiabatic Theorems), (Applied Mathematical Sciences, 72), XI+360 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

It is known that the most of differential equations are not integrable even they do not admit a complete qualitative investigation. However, some equations often can be considered as perturbed systems of simpler equations whose solutions or behaviour of the solutions are known. The perturbation theory reduces the investigation of the system to that of the simpler perturbed equation. If the effect of the perturbation is characterized by a small parameter, then this reduction is made by the method of averaging. It is needed when we are interested in the behaviour of solutions during a long period. For example, in celestial mechanics, the unperturbed model takes into account the interactions between only the Sun and the planets; the interactions between the planets give the perturbations, whose effects are characterized by the proportions of the masses of the planets and the Sun. The method of averaging gives the opportunity of predicting the evolution of the orbits of planets during the thousands of years.

The book is an excellent well-written monograph of the averaging method. It yields a systematic treatment of the modern results, a great part of which was available earlier only in Russian. It gives a good survey on the general case and examines the role of ergodicity in averaging. Special attention is devoted to Hamiltonian systems and to the relation of stability results of the averaging method to Kam Theory. Finally, both classical and quantum adiabatic theorems are considered.

The monograph should be useful to researchers and graduate students in applied mathematics, engineering and physics.

L. Hatvani (Szeged)

Gerhard Maess, Vorlesungen Über numerische Mathematik, II. Analysis, 327 pages, Akademie Verlag, Berlin and Birkhäuser Verlag, Basel, 1988.

This two-volume introduction contains the standard "first book" topics in Numerical Analysis. For orientation here follows a brief review of the contents with a few characteristic keywords in parentheses.

Volume I. Linear Algebra (published separately)

Ch. 1. The basics of numerical computation (roundoff error analysis, interval arithmetic).

Ch. 2. Systems of linear equations (direct and iteration methods, sparse systems, error analysis).

Ch. 3. Overdetermined linear systems (linear least squares, orthogonalization techniques, pseudoinverse).

Ch. 4. Eigenvalue problems (basic facts, vector iteration, QR-method).

Volume II. Analysis

Ch. 5. (Systems of) nonlinear equations (bisection, secant and Newton method, roots of polynomials, inclusion theorems, Bairstow's method, generalized Newton method, rank 1 methods).

Ch. 6. Interpolation and approximation (interpolation by polynomials and spline functions, least square approximations, fast Fourier transform, best approximation with polynomials).
Ch. 7. Numerical quadrature and cubature (Newton—Cotes formulae, Richardson-extrapolation, Gaussian integration methods, Newton—Cotes cubature).
Ch. 8. Initial value problems of ODE (one-step methods, explicit Runge—Kutta methods, extrapolation methods).

The author follows an "inductive Scheme" by expanding each of the subtopics. As a first step, different examples are exposed from various fields, e.g. classical mathematics, physics, engineering, etc. After the mathematical formulation of the problem and the theoretical background the discussion of the algorithm(s) starts. These algorithms are presented in tabular form using "semi-formalized" natural language. A few numerical experiments of the reviewer justify the author's claim: these descriptions can be converted with moderate efforts into working computer programs, indeed. Each topic ends with a "notes and remarks" section that gives a lot of valuable information both on the mathematical details and the bibliographical sources of the relevant methods. Finally, hundreds of exercises are listed ranging from simple calculations to theoretical problems loosely connected with the tackled methods. Moreover, 12 pages on literature (approximately half of them with German items) and a detailed index section aid the reader. This volume contains much more material than you would think at first sight. But this is not a compliment: the typesetting is a bit thick.

The main body of the book contains relatively few full proofs. This makes it suitable for undergraduate courses even for students with neither a math nor a computer science major. The more talented student can find hints for further study at the end of the sections.

J. Virágh (Szeged)

Eli Maor, To Infinity and Beyond, A Cultural History of the Infinite, XVI+275 pages, with 162 illustrations and 6 color plates, Birkhäuser, Boston—Basel—Stuttgart, 1987.

"I see it, but I don't believe it!" (G. Cantor in a letter to R. Dedekind.)

Perhaps you have read the paper of R. Péter "Mathematics is beautiful" published in The Mathematical Intelligencer Vol. 12, No. 1. In this paper R. Péter tried to show that one can fall in love with mathematics, because it is beautiful. It is possible that the succeeded to convince you, but after reading the book of E. Maor you have not the slightest doubt that mathematics is really beautiful, and perhaps you will find more handsomeness in everyday life as well. This book is written for everyone who is eager to know the world over us, for everyone who is curious to catch the beauty around us.

The Infinity is an interesting question for every educated person, because it has many faces. In this book the discussed four faces are: Mathematical Infinity, Geometric Infinity, Aesthetic Infinity, and Cosmological Infinity. (There is an Appendix with some necessary mathematical results.)

Every chapter is a masterpiece, but for me the third chapter on aesthetic infinity was the shocking hit and above all M. Escher Master of the Infinite in it. Let us cite Escher: "I am happy about the contact and friendship of mathematicians that resulted from it all. They have often given me new ideas, and sometimes there even is an interaction between us. How playful they can be, those learned ladies and gentlemen!"

The illustrations, the presentation of the book are marvellous.

In my opinion this work is extraordinarily useful for teachers, after reading it we understand better our mission.

L. Pintér (Szeged)

Mathematical Aspects of Scientific Software, Edited by J. R. Rice (The IMA Volumes in Mathematics and Its Applications, 14), XI+208 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

From the Preface:

"Scientific software is the fuel that drives today's computers to solve a vast range of problems. Huge efforts are put into developing new software, new systems and new algorithms for scientific problem solving. The ramifications of this effort echo throughout science and, in particular, into mathematics. This book explores how scientific software impacts the structure of mathematics, how it creates new subfields and how new classes of mathematical problems arise."

The Editor identifies a dozen of characteristic topics in his opening survey. The 8 further contributions are from the following fields.

1. Different aspects of parallelism are investigated in "The Mapping Problem in Parallel Computation" (by F. Berman) and "Data Parallel Programming and Basic Linear Algebra Subroutines" (by S. L. Johnsson).
2. Symbolic and numeric computation is the content of the works "Scratchpad II: An Abstract Datatype System for Mathematical Computation" (by R. D. Jenks, R. S. Sutor and S. M. Watt) and "Integrating Symbolic, Numeric and Graphics Computing Techniques" (by P. S. Wang).
3. In the paper "Performance of Scientific Software" (by E. N. Houstis, J. R. Rice, C. C. Christara and E. A. Vavalis) an abstract Performance Evaluation Model is introduced and applied for elliptic PDE solvers of the ELLPACK.
4. The remaining three works are more or less connected with algorithms for geometry: "Applications of Gröbner Bases in Non-linear Computational Geometry" (by B. Buchberger), "Geometry in Design: The Bezier Method" (by G. Farin) and C. M. Hoffmann's "Algebraic Curves".

Although the contributions are rather different both in scope and depth (my favorites were Buchberger's paper and the Scratchpad II description), as a whole, they give a good overview of this rapidly developing branch.

J. Virágh (Szeged)

Mathematical Economics, Edited by A. Ambrosetti, F. Gori and R. Lucchetti (Lecture Notes in Mathematics, 1330), VII+137 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

Nowadays more and more sciences have used mathematics. This phenomenon results in very fruitful interactions between mathematics and sciences, among them economics. The varying relationship between mathematicians and economists has been described very properly by the first paragraph of Introduction of the book: "In the last few years an ever increasing interest has been shown by economists and mathematicians in deepening and multiplying the many links already existing between their areas of research. Economists are looking for more advanced mathematical techniques to be applied to the analysis of formal models of greater complexity; mathematicians have found in problems from economics the stimulus to start new directions of study and to explore different trends within their theories."

To offer scholars from the two fields an opportunity of meeting and working together, the Centro Internazionale Mathematico Estivo (C.I.M.E.) organized a Session on "Mathematical Economics" at Villa La Querceta in Montecatini Terme, Italy in 1986. The present lecture notes contains the subject-matter of the four survey courses of the Session: I. Ekeland, Some variational methods arising from mathematical economics; A. Mas-Cobell, Four lectures on the differentiable approach to general equilibrium theory; J. Scheinkman, Dynamic general equilibrium models; S. Zamir, Topics in non-cooperative game theory.

After having read the volume one believes that the editors' opinion is true: the mathematics has become more and more important for economists, and the mathematicians can find more and more stimulating problems in economics.

L. Hatvani (Szeged)

László Máté, Hilbert Space Methods in Science and Engineering, VIII+273 pages, Akadémiai Kiadó, Budapest, 1989.

Do you think that you ought to become more familiar with mathematical methods applied in your special line? Certainly many students, engineers and scientists do so. One, who uses methods based on "higher" mathematics, has often not enough time and energy for studying such disciplines.

In case if this discipline is Hilbert space theory, the present book can certainly help to overcome the difficulties. Even a look at the table of "Contents" shows that it contains pretty large material, the potential reader surely will find something of his special interest.

Reading then only a few pages, one can easily recognize: This book differs from a usual "Introduction to the Theory of ...". This is a work for those, who want to understand the basic facts and notions, to get aquainted with the corresponding methods, but prefers some illustrative examples rather than the details of complicated arguments. However, the proofs are not completely omitted, and the book is far from a collection of definitions, theorems and simple descriptions of methods to be applied. From the Preface: "The bulk of the applications revolve around reproducing kernel Hilbert spaces and causal operators. Several applications are treated here for the first time at an introductory level."

The prerequirements certainly will not keep back the reader from understanding the text. He needs to be familiar only with the elements of mathematical analysis and basic facts of linear algebra. Then how to treat L^2-spaces? Well, the author considers only the submanifold consisting of the continuous functions whenever he can do so. However, the concept of Lebesgue integral clearly cannot be completely avoided.

Who to recommend this book to? I think, the answer is suggested already by its title.

E. Durszt (Szeged)

Johannes C. C. Nitsche, Lectures on Minimal Surfaces Vol. 1, XXV+563 pages, Cambridge University Press, New York—New Rochelle—Melbourne—Sydney, 1989.

This book is the enlarged and updated version of the first five chapters of the author's book "Vorlesungen über Minimalflachen" originally published in the series "Grundlehren der matematischen Wissenschaften" in 1975.

Since the last decade has been one of extraordinary research activities on all fronts of minimal surface theory recently there was a claim to a more or less synthetic book of this subject. In the last decade the key book of this topic was the author's one, mentioned above, which not only summarized the results before but also formulated research problems (mainly in section IX. 2) which have become actively studied.

Therefore I could not find better solution to satisfy the real titles of the subject than this development of the German book.

We avoid the detailed review of the book because its original is so well known. It preserved the spirit and scope of the Vorlesungen. Its style with many figures is as clear as or sometimes clearer than its original version. We are looking forward the second and third parts.

In our opinion this book is indispensable to anyone working in the field.

Á. Kurusa (Szeged)

R. S. Palais—C. L. Terng, Critical Point Theory and Submanifold Geometry (Lecture Notes in Mathematics, 1353), X+271 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1988.

This book grew out of the lectures given by the authors at Nankai Institute of Mathematics, Tianjin, China in May of 1987. The book consists of two, inter-related parts corresponding to the two series of these lectures. Chapters 1—4 of Part I give an introduction to the classical and modern submanifold theory of Euclidean and Hilbert spaces: Levi—Civita connections, vector fields and differential equations, local invariants of submanifolds, fundamental theorem of submanifolds in space forms, Weingarten surfaces in three dimensional space forms, immersed flat tori in S^3, focal structure of submanifolds. Chapters 5—8 contain a systematic treatment of the theory of isoparametric submanifolds of Hilbert spaces developed by the authors in the last years. (A submanifold is called isoparametric if its normal curvature is zero and the principal curvatures along any parallel normal field are constant.) These submanifolds arise naturally in representation theory for, in particular the principal orbits of the isotropy representation of a symmetric space are homogeneous isoparametric, but there are also many non-homogeneous examples. Part II is a self-contained account of critical point theory on Hilbert manifolds. The two parts are connected through the Morse Index Theorem, which is applied to the investigation of the topology of isoparametric submanifolds of Hilbert spaces.

The reader is assumed to be familiar only with the elementary theory of differentiable manifolds and Riemannian geometry. The book is a very good introduction to the research problems in the field, and can be warmly recommended to the mathematicians who are interested in a beautiful interplay between Riemannian geometry, functional analysis, topology and transformation group theory.

Peter T. Nagy (Szeged)

R. R. Phelps, Convex Functions, Monotone Operators and Differentiability (Lecture Notes in Mathematics, 1364), VII+114 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

This clearly written book contains wealthy material on the differentiability properties of convex functions on infinite dimensional spaces. The genesis of this work is the set of notes "Differentiability of convex functions on Banach spaces" which was written by the author for a graduate course at University College London.

The first Chapter deals with the definition of convex functions; Mazur's theorem on differentiability of convex function on a separable Banach space; and the subdifferential of a convex function. Chapter 2 is devoted to the monotone operators (as a matter of fact the subdifferential is a special case of a monotone operator). The results of this Chapter all involve continuous convex functions defined on open convex sets. But in many applications lower semicontinuous convex function should be considered. This topic is the main subject of Chapter 3. Borwein—Preiss smooth-variational principle (which uses differential perturbation) is the central question in Chapter 4. The following Chapter 5 deals with spaces with Radon—Nikodym property and some questions related to the optimization (particularly to the so-called perturbed optimization). The short Chapter 6 is devoted to the class of Banach spaces in which every continuous convex function is Gateaux differentiable in a dense set of points. Chapter 7 gives a generalization of monotone operators to the upper semicontinuous compact valued maps.

The present book is highly recommended to all who interested in mathematical analysis.

J. Németh (Szeged)

András Recski, Matroid Theory and its Applications, XIII+531 pages, Akadémiai Kiadó, Budapest, 1989.

Matroid theory is one of the most deepest parts of combinatorics, as well as being one of the most important ones from the point of view of application.

The author's aim was to show the present state of the theory with special emphasis on its algorithmic aspects and on the applications in electrical engineering and in statics.

The book is divided into two parts. The first part contains the background about graphs and algebra.

Here it can be found the fundamental constructions and algorithms in graph theory which serve as a root of matroids: trees, forests, cut sets, circuits, planar graphs, duality, matching for bipartite graphs and flow theory.

The second part is devoted to matroids. After presenting the basic concepts such as duality, minors, direct sum, connectivity, greedy algorithm, the following topics are discussed in detail: the representations of matroids, the sum of matroids, induced matroids. The last "mathematical" section is devoted to some recent results in matroid theory.

The book is self-contained, written very carefully. Mathematical results are presented in the odd numbered chapters only. Even numbered chapters describe associated applications. It contains quite a lot of figures which illustrate the abstract constructions and proofs. Each section ends with exercises for checking the reader's understanding and problems which are more difficult.

The book is recommended to both mathematicians and engineers who are interested in matroid theory and its applications but it may serve as a handbook for researchers in matroid theory.

J. Kincses (Szeged)

Elmer G. Rees, Notes on Geometry (Universitext), VIII+109 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1988.

This book is about the Euclidean, the projective and the hyperbolic geometry. In some sense it is a review on these topics from a relatively new viewpoint. At the same time it uses the material most natural for undergraduate students, such as linear algebra, group theory, metric spaces and complex analysis.

What is new in this book is its viewpoint, which makes bridge between the classical geometry and the topology or differential geometry. While this viewpoint gives undergraduate students a deeper understanding of these geometries it may be considered as a step before the first one to differential geometry.

We recommend this book to all undergraduate students interested in geometry, because of the advantages mentioned above and the concrete treats of these very important geometries. The students can also find a large number of exercises and problems.

Á. Kurusa (Szeged)

Martin Schlichenmaier, An Introduction to Riemann Surfaces, Algebraic Curves and Moduli Spaces (Lecture Notes in Physics, 322), VII+148 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1989.

The algebraic and analytic geometry are becoming more and more useful in theoretical physics due to the string theories among others. This book contains an introductory lecture course for physicists in this subjects.

Although the book is very thin one may be surprised about what a large number of topics are in it. At the same time, and this is really a surprise, it can teach basic calculations together with a basic view of these subjects. This book is not self-contained because some fundamental theorems, as the Riemann—Roch theorem for example, are not proven, but I think this makes it more useful as an introduction for physicists. If a reader would like to learn further the well-selected bibliography will help him/her.

In sum, we recommend this book to all physicists who want to know these modern subjects and to students of mathematics or physics who are looking for a short introduction to the various aspects of these subjects.

Á. Kurusa (Szeged)

L. Sirovich, Introduction to Applied Mathematics (Texts in Applied Mathematics, 1) XII+370 pages, Springer-Verlag, New York—Berlin—Heidelberg—London—Paris—Tokyo, 1988.

It is a commonplace that mathematics plays a more and more important role in various sciences. The circle of the users of mathematics is continuously widening. The aim of this series is to present textbooks for use in advanced undergraduate and beginning graduate courses.

Perhaps the main problem of this kind of works is in some sense pedagogical: to give the students confidence in mathematical methods and — the most important — success in problem solving. The appropriate rank of mathematical rigor in discussions makes the work understandable and applicable.

This book grew out of courses held by the author at Brown University. Chapter headings are: Complex Numbers, Convergence and Limit, Differentiation and Integration, Discrete Linear Systems, Fourier Series and Applications, Spaces of Functions, Partial Differential Equations, The Fourier and Laplace Transforms, Partial Differential Equations (continued).

The author's basic aim was to present standard methods, as he says "a basic bag of tricks". He disregards the theorem-proof format for the sake of a more informal style.

The distinguishing feature of the work is the 4th chapter on discrete linear systems. Here one finds periodic sequences; discrete Fourier series and transforms; the fast Fourier algorithm; the cell model of diffusion; the Z-transform and applications; the Wiener—Hopf method.

Two chapters are devoted to the partial differential equations applying various methods especially an eigenfunction approach. Perhaps a little more discussion on the Dirac delta function would be useful.

There is a fine collection of exercises, in which many important applications appear. Perhaps some more hints would be useful but separately and not immediately after the problems.

At last I would like to mention the extraordinarily beautiful and useful illustrations.

L. Pintér (Szeged)

Hans L. Trinkaus, Probleme? Höhere Mathematik! Eine Aufgabensammlung zur Analysis, Vektor- und Matrizenrechnung, Herausgegeben von H. Neunzert (Mathematik für Physiker und Ingenieure), IX+337 pages, Springer-Verlag, Berlin—Heidelberg—New York—London—Paris—Tokyo, 1988.

This is an interesting book and I would like to have one on my table.

The book consists of two approximately equal parts: 1. Theory and Praxis, 2. Results (solutions and remarks). Some titles from the first part: real numbers, mathematical induction, complex numbers, real functions, sequences, series, differentiation, integration, Taylor series, R^n space, matrices, determinants. In every chapter you will find the necessary simple notations, clear definitions, important theorems (without proofs), several interesting remarks, and then the most essential part: the exercises. These are useful both for students and teachers, because almost every problem is in strong connection with everyday life. There are problems from physics, biophysics, chemistry, biology, psychology, music, sports (e.g. Fosbury-Flop) and so on.

In studying physics (or more general natural sciences) students have troubles because they do not know the necessary mathematical ideas or they cannot apply their mathematical knowledge. (Sometimes they believe that it is good for nothing.) This book can change their ground. The literary citations (the authors are: Goethe, Platon, Kant, Nietzsche, Storm, Camus and others) and above all the historical remarks (quotations, letters, dates etc.) make this work a many-coloured reading.

Finally here is one of the inspiring citations (it may be found in chapter on mathematical induction): "Man heisst die Ehe gut, erstens weil man sie noch nicht kennt, zweitens weil man sich an sie gewöhnt hat, drittens weil man sie geschlossen hat, — das heisst fast in allen Fallen. Und doch ist damit nichts für die Güte der Ehe überhaupt bewiesen." (F. Nietzsche).

L. Pintér (Szeged

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